SOME PROPERTIES OF LINE GRAPHS OF HAMMING AND JOHNSON GRAPHS

By

Miss Ratinan Chinda

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science Program in Mathematics

Department of Mathematics

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สมบัติบางประการของกราฟเส้นของกราฟแฮมมิ่งและกราฟจอห์นสัน

โดย
นางสาวรตินันท์ จินดา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์
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The Graduate School, Silpakorn University has approved and accredited the Thesis title of "Some properties of line graphs of Hamming and Johnson graphs" submitted by Miss Ratinan Chinda as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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53305205 : MAJOR : MATHEMATICS
KEYWORDS : HAMMING GRAPHS/ JOHNSON GRAPHS/ LINE GRAPHS/ CONNECTIVITY/ EDGE CHROMATIC NUMBER

RATINAN CHINDA : SOME PROPERTIES OF LINE GRAPHS OF HAMMING AND JOHNSON GRAPHS. THESIS ADVISOR : CHALERMPONG WORAWANNOTAI, Ph.D. 33 pp.

In this thesis, we show that the connectivity of the line graph of a Hamming graph or Johnson graph with degree k is $2k-2$. Moreover, we characterize the graphs whose line graphs are either Hamming graphs or Johnson graphs and provide the chromatic numbers of the line graphs of Hamming graphs.

Department of Mathematics Graduate School, Silpakorn University

Student's signature .... Academic Year 2013

Thesis Advisor's signature ....

53305205 : สาขาวิชาคณิตศาสตร์
คำสำคัญ : กราฟแฮมมิ่ง / กราฟจอห์นสัน / กราฟเส้น / ค่าจุดเชื่อมโยง / การระบายสีเส้นเชื่อม

รตินันท์ จินดา : สมบัติบางประการของกราฟเส้นของกราฟแฮมมิ่งและกราฟจอห์นสัน อาจารย์ที่ปรึกษาวิทยานิพนธ์ : ดร. เฉลิมพงศ์ วรวรรโณทัย. 33 หน้า.

ในวิทยานิพนธ์นี้ เราแสดงว่ากราฟเส้นของกราฟแฮมมิ่งและกราฟจอห์นสันที่มีดีกรีของจุดในกราฟคือ k จะมีค่าจุดเชื่อมโยงเป็น $2k-2$ นอกจากนั้นเรายังได้จำแนกกราฟที่มีกราฟเส้นเป็นกราฟแฮมมิ่งหรือกราฟจอห์นสัน และยังบอกถึงรงคเลขของกราฟเส้นของกราฟแฮมมิ่งด้วย

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ลายมือชื่อนักศึกษา.. ปีการศึกษา 2556

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

ACKNOWLEDGEMENTS

This thesis ha been completed by the involvement of people about whom I would like to mention here.

I would like to thank Dr. Chalermpong Worawannotai, my advisor for his valuable suggestions and excellent advices throughout the study with great attention.

I would like to thank Assoc. Prof. Dr. Nawarat Ananchuen and Asst. Prof. Dr. Thiradet Jiarasuksakun, chairman and member of the thesis committee, for their valuable comments and suggestions.

Finally, I would like to express my gratitude to my family and my friends for their understanding, encouragement and moral support during the study.

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Chapter 1

Introduction

In this chapter we recall some definitions and notations used in this thesis.

A *graph* is an ordered pair $G = (V(G), E(G))$ where $V(G)$ is a nonempty set of vertices and $E(G)$ is a set of unordered pairs of vertices. The elements of $E(G)$ are called *edges*. If $\{u, v\} \in E(G)$, we denote $\{u, v\}$ by uv and we say that u and v are *adjacent*. For any $e = uv \in E(G)$, the vertices u and v are called the *end vertices* of uv, and e is said to *join* u and v. Two edges are *adjacent* if they have a common end vertex.

A graph is said to be *finite* if it has finite vertices and edges. An *empty* graph is a graph with no edges. The *order* of a graph is the number of its vertices. An edge with identical end vertices is called a *loop*. Two or more edges that join the same pair of vertices are called *parallel edges*. A graph with no loops nor parallel edges is called a *simple graph*. All graphs considered in this thesis are finite and simple.

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say G is *isomorphic to* H and write $G \cong H$ if there is an isomorphism from G to H. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a nonempty subset S of $V(G)$, the *subgraph of* G *induced* by S, is the graph $G[S]$ with $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G)|u, v \in S\}$. A *spanning subgraph* of G is a subgraph of G which contains every vertex of G. A *neighbor* of a vertex v in G is a vertex which is adjacent to v and the *neighborhood* of v in G, denoted by $N_G(v)$, is the set $\{u \in V(G)|uv \in E(G)\}$. The *degree* of v in G is $|N_G(v)|$. The minimum degree of all vertices in G is denoted by $\delta(G)$ and the maximum degree of all vertices in G is denoted by $\Delta(G)$.

A *complete graph* is a graph in which every pair of vertices are adjacent. A complete graph of order n is denoted by K_n. A *clique* of a graph is a complete subgraph which is not contained in any other complete subgraphs. A graph G is *bipartite* if $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. Moreover, if every vertex of V_1 is joined to every vertex of V_2, then G is called a *complete bipartite* graph, denoted by $K_{m,n}$ where $m = |V_1|$ and $n = |V_2|$.

A *walk* of length n from a vertex u to a vertex v in a graph is a finite sequence $(u = v_0, v_1, ..., v_{n-1}, v_n = v)$ such that any two successive vertices are adjacent; when there is no ambiguity, we may drop the parentheses. The vertices $v_1, ..., v_{n-1}$ are called the *internal vertices* of the walk. A *path* is a walk with distinct vertices. A path from a vertex u to a vertex v is denoted by $u - v$. If a path $u - v$ and a path $v - w$ have only v as a common vertex, then they can be concatenated to be a path $u - w$ and we denote this concatenated path by $u - v - w$. A *cycle* is a walk that starts and ends at the same vertex but otherwise has distinct vertices. A cycle with n vertices is denoted by C_n. Two vertices u and v are *connected* if there is a walk from u to v. A graph G is *connected* if every pair of its vertices are connected; otherwise G is *disconnected*. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in G. The *diameter* of G, denoted by $diam(G)$, is the maximum distance between two vertices of G.

The *(vertex-)connectivity* of a graph G, denoted by $\kappa(G)$, is the least number of vertices whose removal (along with all incident edges) disconnects G or reduces it to an empty graph. The *edge-connectivity* of G, denoted by $\lambda(G)$, is the least number of edges whose removal disconnects G or reduces it to an empty graph. A graph G is called *n-connected* if $\kappa(G) \geq n$ and *m-edge connected* if $\lambda(G) \geq m$. For two distinct vertices u and v of a graph G, a set of $u - v$ paths is said to be *internally disjoint* if u and v are the only vertices that any pair of distinct paths in the set have in common. We will write *id-paths* instead of internally disjoint paths. A set of two or more paths is said to be *disjoint (edge-disjoint)* if there is no common vertex (edge) in any pair of paths in the set. We will write *ed-paths* instead of edge-disjoint paths.

A *regular graph* is a graph where each vertex has the same degree. This common degree is the *degree* of the graph. A *k-regular graph* is a regular graph with degree k. A connected graph G is *distance-regular* if it is a regular graph and there exist integers b_i, c_i for $i \in \{0, ..., diam(G)\}$ such that for any two vertices x, y in G and distance $i = d(x, y)$, there are exactly c_i neighbors of y in $G_{i-1}(x)$ and b_i neighbors of y in $G_{i+1}(x)$, where $G_j(x)$ is the set of vertices z of G with $d(x, z) = j$. Let S be a set of n elements and let d be a positive integer. A *Hamming graph* $H(d, n)$ is a graph whose vertex set is the set of words of length d of elements of S and two vertices are adjacent if they differ in precisely one position. The set S of the Hamming graph is called an *alphabet set*.

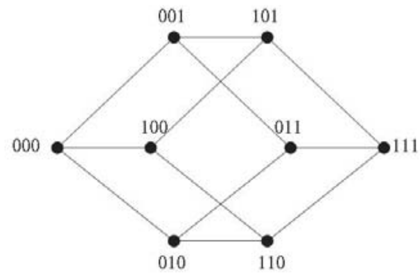


Figure 1 : Hamming graph $H(3, 2)$ where {0,1} is its alphabet set.

A *Johnson graph* $J(n, m)$, where $m \in \{0, 1, ..., n\}$ is a graph whose vertex set is the set of m-subsets of S and two vertices are adjacent if they have exactly $m - 1$ elements in common. The set S of Johnson graph is also called an alphabet set.

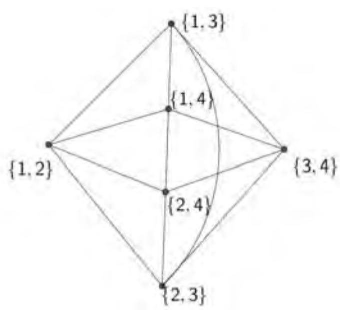


Figure 2 : Johnson graph $J(4, 2)$ where {1,2,3,4} is its alphabet set.

A Hamming graph $H(d,n)$ is distance-regular with $c_i = i$ and $b_i = (n-1)(d-i)$, for $i = 0, ..., diam(H(d,n))$. A Johnson graph J(n,m) is distance-regular with $c_i = i^2$ and $b_i = (m-i)(n-m-i)$, for $i = 0, ..., diam(J(n,m))$.

Let G be a nonempty graph. The *line graph* of G, denoted by $L(G)$, is the graph whose vertex set is the edge set of G and two vertices of $L(G)$ are adjacent if and only if they are two adjacent edges in G. A graph H is called a *line graph* if there exists a graph F such that $H \cong L(F)$. The *star* at a vertex v in G is the set of all edges incident to v. A nonempty subgraph X of G is said to *induce* a subgraph Y of $L(G)$ if Y is the subgraph of $L(G)$ induced by the vertices in $E(X)$. In particular, a nontrivial path in G induces a path in $L(G)$ and a nonempty star in G induces a clique in $L(G)$.

A collection $\mathcal{K}$ of subgraphs of a graph G is called a *Krausz partition* of G if it satisfies the following three properties:

(1) each member of $\mathcal{K}$ is a complete graph;

(2) every edge of G is in exactly one member of $\mathcal{K}$;

(3) every vertex of G is in exactly two members of $\mathcal{K}$.

For a graph G, the least number of colors needed to color the vertices of G in such a way that no two adjacent vertices have the same color is called the *chromatic number* of G, denoted by $\chi(G)$. Similarly, the least number of colors needed to color the edges of G in such a way that no two adjacent edges have the same color is called the *edge-chromatic number* of G, denoted by $\chi^1(G)$. Observe that $\chi(L(G))$ is $\chi^1(G)$.

For a graph G, a *k-factor* of G is a spanning k-regular subgraph of G and G is said to be *k-factorable* if the edges of G can be partitioned into edge disjoint k-factors. For $M \subseteq E(G)$, M is a *matching* in G if no two edges of M are adjacent; the two end vertices of each edge of M are said to be *matched* under M.

For a graph G, Whitney [8] showed that $\kappa(G) \leq \lambda(G) \leq \delta(G)$. For a distance-regular graph G with degree k, Brouwer and Koolen [4] showed that $\kappa(G) = k$ and thus we have $\kappa(G) = \lambda(G) = k$.

For a line graph of a distance-regular graph, we conjecture that their connectivity, edge-connectivity and degree are equal as well. In this thesis we prove our conjecture for two families of distance-regular graphs which are Hamming graphs and Johnson graphs.

Moreover, we provide the characterization of the graphs whose line graphs are Hamming graphs and Johnson graphs and provide the chromatic numbers of line graphs of Hamming graphs.

Chapter 2

Preliminaries

In this chapter we state a number of results that we will use.

The following are results on connectivity and edge-connectivity.

Theorem 2.1. [8, Theorem 5] For any graph G, we have $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 2.2. [4] Let G be a non-complete distance-regular graph with degree $k > 2$. Then $\kappa(G) = k$.

Theorem 2.3. [6, Theorem A] A graph G is n-connected (m-edge connected) if and only if there exist at least n id-paths (m ed-paths) between every pair of distinct vertices.

Remark 2.4. Ed-paths in a graph G induce disjoint paths in $L(G)$.

Theorem 2.5. [1,Exercise 3.2.3] Let G be an n-connected graph and let S_1 and S_2 be subsets of $V(G)$ such that $1 \leq |S_1| = |S_2| = m \leq n$. Then there exists m disjoint paths $P_1, P_2, ..., P_m$ such that P_i is a path from a vertex in S_1 to a vertex in S_2 for all $i \in \{1, ..., m\}$.

The following are results on line graphs.

Theorem 2.6. [2, Theorem 2.1(ii)] If a graph G is k-regular, then $L(G)$ is $(2k-2)$-regular.

Theorem 2.7. [6, Corollary 2a] For any graph G, we have $\lambda(L(G)) \geq 2\lambda(G) - 2$.

Theorem 2.8. [2, Theorem 2.1(vii)] If G is a connected graph with n vertices, where $n \neq 3$, then $L(G) \cong K_n$ if and only if $G \cong K_{1,n}$.

Theorem 2.9. [2, Theorem 4.3] A graph G is not a line graph if G has $K_{1,3}$ as an induced subgraph.

Theorem 2.10. [2, Corollary 4.2] Let H be a connected line graph of a graph F. Suppose H is not one of the graphs $L(G_i)$ in Figure 3, then H has one and only one Krauz partition. Moreover, all members of this partition are induced by the stars at the vertices of F.

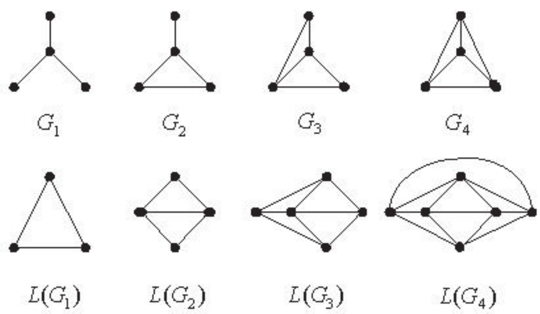


Figure 3 : The graphs whose line graphs have more than one Krauz partition.

The following are results on edge-chromatic numbers.

Theorem 2.11. [7, Theorem 6.14] (Vizing's Theorem) For any graph G,

$$\Delta(G) \leq \chi^1(G) \leq \Delta(G) + 1.$$

Theorem 2.12. [5, p.302] For a k-regular graph G with $k \geq 1$,

$$\chi^1(G) = k \text{ if and only if } G \text{ is 1-factorable.}$$

Theorem 2.13. [1, Theorem 4.2.2(3)] For any positive integer n, the complete graph K_{2n} is 1-factorable.

Chapter 3

Line graphs of Hamming graphs

In this chapter we present the results on the connectivity of line graphs of Hamming graphs. In addition we characterize the graphs whose line graphs are Hamming graphs and provide the chromatic numbers of line graphs of Hamming graphs.

Theorem 3.1. For a distance-regular graph G with degree k, we have $\lambda(L(G)) = 2k-2$.

Proof. By Theorem 2.1, Theorem 2.2, Theorem 2.6 and Theorem 2.7. □

Lemma 3.2. For a k-regular graph G, we have $\kappa(L(G)) \leq 2k-2$.

Proof. By Theorem 2.1 and Theorem 2.6. □

Lemma 3.3. Let G be a distance-regular graph with degree k. Then there exist $2k-2$ id-paths between any two adjacent vertices in $L(G)$.

Proof. If $k=1$, then G is K_2 and $L(G)$ is K_1. There is nothing to prove here.

If $k=2$, then G is a cycle since a connected 2-regular graph is a cycle. Therefore, $L(G)$ is a cycle and there exist $2k-2=2$ id-paths between any two adjacent vertices in $L(G)$.

Now suppose $k \geq 3$. Let e_A and e_B be two adjacent vertices in $L(G)$. Then they are two adjacent edges in G. Let u_A, u_B and v be the vertices of G such that $e_A = u_A v$ and $e_B = u_B v$. Then there exist $k-2$ more edges which are incident with v. Let these edges be $e_1, e_2, ..., e_{k-2}$. Then $P := \{(e_A, e_i, e_B) | 1 \leq i \leq k-2\} \cup \{(e_A, e_B)\}$ is a set of $k-1$ id-paths from e_A to e_B in $L(G)$.

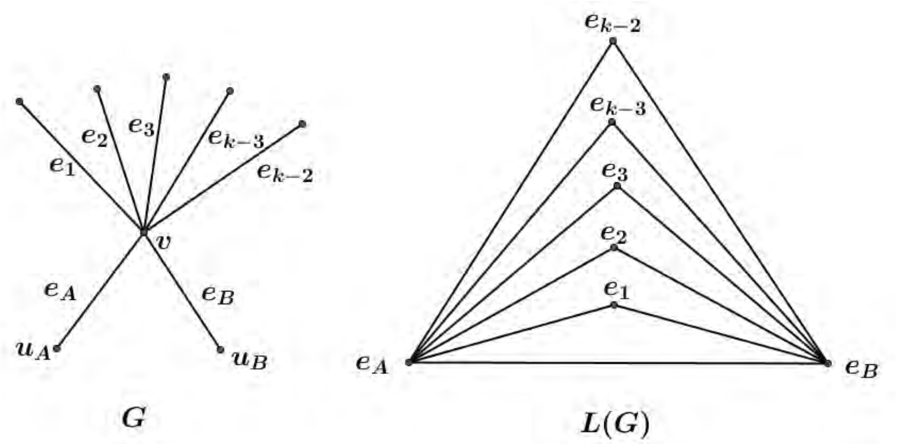


Figure 4 : $k-1$ id-paths from e_A to e_B in $L(G)$.

Since G is distance-regular with degree $k \geq 3$, by Theorem 2.2 we have $\kappa(G) = k$ and thus by Theorem 2.3, there exist k id-paths from u_A to u_B in G. Let Q be the set of these k paths. Since G is k-regular and v is a common neighbor of u_A and u_B, the set Q contains the path (u_A, v, u_B). By this and since the paths in Q are id-paths, none of $e_1, e_2, ..., e_{k-2}$ is in any paths of Q. Therefore,

$Q \setminus \{(u_A, v, u_B)\}$ is the set of $k-1$ id-paths from u_A to u_B in G.

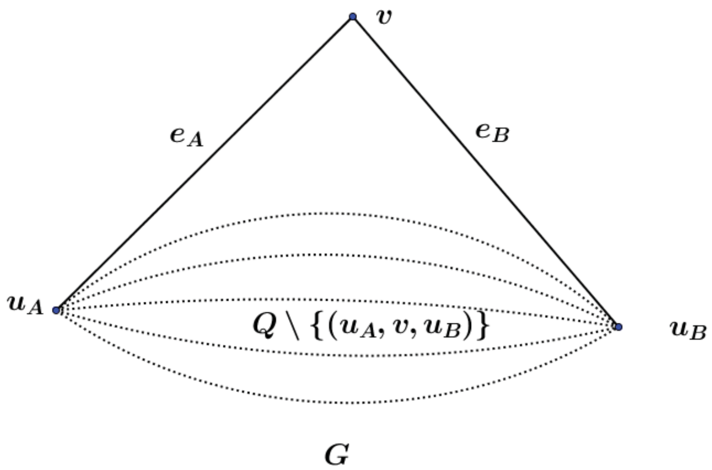


Figure 5 : $k-1$ id-paths in $Q \setminus \{(u_A, v, u_B)\}$ from u_A to u_B in G

These paths induce $k-1$ id-paths from e_A to e_B in $L(G)$ which differ from the paths in P. Hence we have $(k-1)+(k-1) = 2k-2$ id-paths from e_A to e_B in $L(G)$ □

Proposition 3.4. Let G be a complete graph with $k+1$ vertices. Then $\kappa(L(G)) = 2k-2$.

Proof. Let e_A and e_B be two distinct edges of G. To show that $\kappa(L(G)) = 2k-2$, by Theorem 2.3 and Lemma 3.2, it suffices to display $2k-2$ id-paths from e_A to e_B in $L(G)$.

If e_A and e_B are adjacent, then there are $2k-2$ id-paths from e_A to e_B in $L(G)$ by Lemma 3.3. Now suppose that e_A and e_B are not adjacent. Observe that ed-paths from the end vertices of e_A to the end vertices of e_B in G induce id-paths from e_A to e_B in $L(G)$. To finish the proof we will display such $2k-2$ ed-paths in G.

Let u_1, u_2, u_3 and u_4 be the vertices such that $e_A = u_1u_2$ and $e_B = u_3u_4$ and let $w_1, w_2, ..., w_{k-3}$ be the remaining vertices in G. Then the following sequences are $2k-2$ ed-paths from the end vertices of e_A to the end vertices of e_B in G:

(u_1, u_3); (u_2, u_4); (u_1, u_4); (u_2, u_3);
(u_1, w_i, u_3), $1 \le i \le k-3$; and
(u_2, w_i, u_4), $1 \le i \le k-3$.

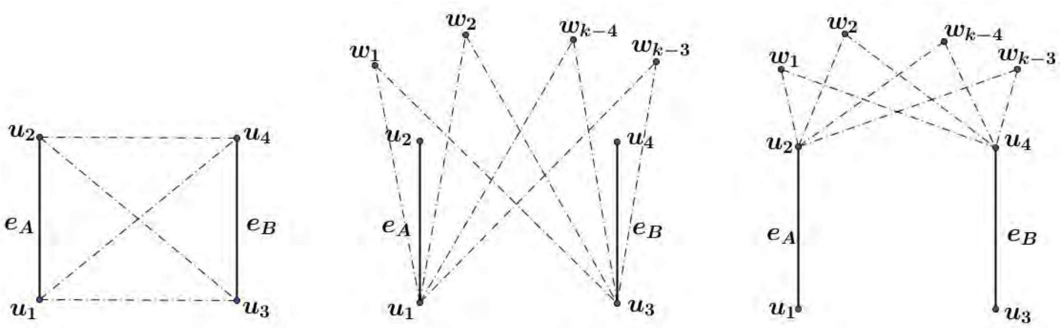


Figure 6 : $2k-2$ ed-paths from the end vertices of e_A to the end vertices of e_B.

□

Theorem 3.5. Let G be a Hamming graph $H(d, n)$ with degree $k = d(n-1)$. Then $\kappa(L(G)) = 2k - 2$.

Proof. Let S be the alphabet set of G.

If $d = 1$, then G is a complete graph and the result is done by Proposition 3.4. Therefore, we may assume that $d \geq 2$. Let e_1 and e_2 be two distinct edges in G. Then e_1 and e_2 are two distinct vertices in $L(G)$. By Theorem 2.3 and Lemma 3.2, to show that $\kappa(L(G)) = 2k-2$ it suffices to display $2k-2$ id-paths from e_1 to e_2 in $L(G)$.

If e_1 and e_2 are adjacent, then there exist $2k-2$ id-paths from e_1 to e_2 in $L(G)$ by Lemma 3.3. Now suppose that e_1 and e_2 are not adjacent. Observe that ed-paths from the end vertices of e_1 to the end vertices of e_2 in G induce id-paths from e_1 to e_2 in $L(G)$. To finish the proof we will display such $2k-2$ ed-paths in G.

For any $u \in V(G)$, $a \in S$ and $1 \leq i \leq d$, let $rp(u, a, i)$ denote the vertex obtained from replacing the element at the position i of the vertex u with a.

For convenience, we write $u - P - v$ to denote path P from vertex u to vertex v. If a $u - v$ path has length 1, then we write $u \sim v$.

For $i = 1, 2$, let z_i be the position where the end vertices of e_i differ. Then either $z_1 = z_2$ or $z_1 \neq z_2$.

Case 1 : $z_1 = z_2$.

Without loss of generality, assume that z_1 is the first position.

Let $a_1, a_2, b_1, b_2, x_1, ..., x_{d-1}, y_1, ..., y_{d-1}$ be the elements of S such that $a_1x_1...x_{d-1}$ and $b_1x_1...x_{d-1}$ are the end vertices of e_1, and $a_2y_1...y_{d-1}$ and $b_2y_1...y_{d-1}$ are the end vertices of e_2.

For any $a \in S$, let H_a be the induced subgraph of G whose vertices have the element a at the position z_1. Then for distinct $a, b \in S$, we have $V(H_a) \cap V(H_b) = \phi$. For any $a \in S$, the subgraph H_a is isomorphic to $H(d-1, n)$. Thus, for any $a \in S$, $\kappa(H_a) = (d-1)(n-1)$ by Theorem 2.2, and for any distinct vertices $u_1, u_2 \in H_a$, there exist $(d-1)(n-1)$ id-paths from u_1 to u_2 in H_a by Theorem 2.3.

For any $a \in S$ and $u \in N_{H_a}(ax_1...x_{d-1})$, let Pa be a set of $(d-1)(n-1)$ id-paths from $ax_1...x_{d-1}$ to $ay_1...y_{d-1}$ in H_a, let $Pa(1), Pa(2)$ be two distinct paths in Pa, let $Pa(u)$ be the path in Pa which contains u, and let $Pa'(u)$ be the path obtained by deleting the vertex $ax_1...x_{d-1}$ from $Pa(u)$.

Let $s_{a_1,1}, s_{a_1,2}, ..., s_{a_1,(d-1)(n-1)}$ be the distinct vertices in $N_{H_{a_1}}(a_1x_1...x_{d-1})$. For any $c \in S$, let $s_{c,1}, s_{c,2}, ..., s_{c,(d-1)(n-1)}$ be the distinct vertices in $N_{H_c}(cx_1...x_{d-1})$ such that $s_{c,i} = rp(s_{a_1,i}, c, z_1)$ for $i \in \{1, ..., (d-1)(n-1)\}$. Then for any distinct $b, c \in S$, the vertex $s_{b,i}$ is adjacent to the vertex $s_{c,i}$ for each $i \in \{1, ..., (d-1)(n-1)\}$.

Case 1.1 : $\{a_1, b_1\} = \{a_2, b_2\}$.

Without loss of generality, assume that $a_1 = a_2$ and $b_1 = b_2$. Then we have $2(d-1)(n-1)$ ed-paths in $Pa_1 \cup Pb_1$.

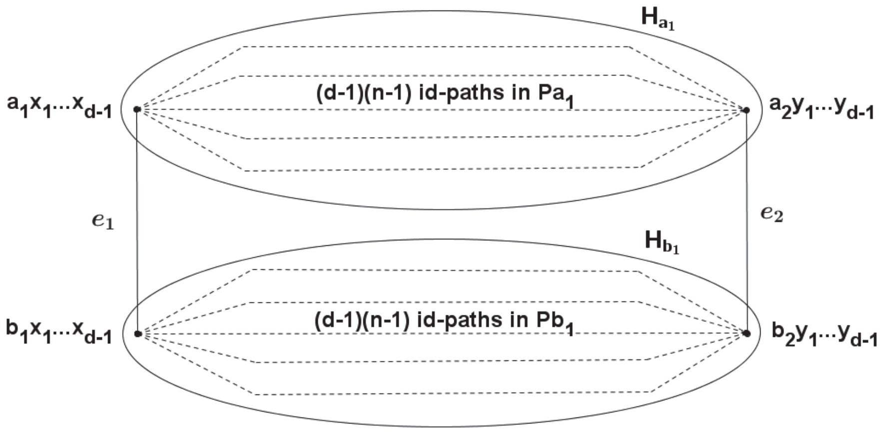


Figure 7 : ed-paths in $Pa_1 \cup Pb_1$

Let $S' = S \setminus \{a_1, b_1\}$. Then $|S'| = n - 2$. Let Q be the set of the following paths:

$a_1x_1...x_{d-1} \sim cx_1...x_{d-1} - Pc(1) - cy_1...y_{d-1} \sim a_2y_1...y_{d-1}$; and

$b_1x_1...x_{d-1} \sim cx_1...x_{d-1} - Pc(2) - cy_1...y_{d-1} \sim b_2y_1...y_{d-1}$

where $c \in S'$.

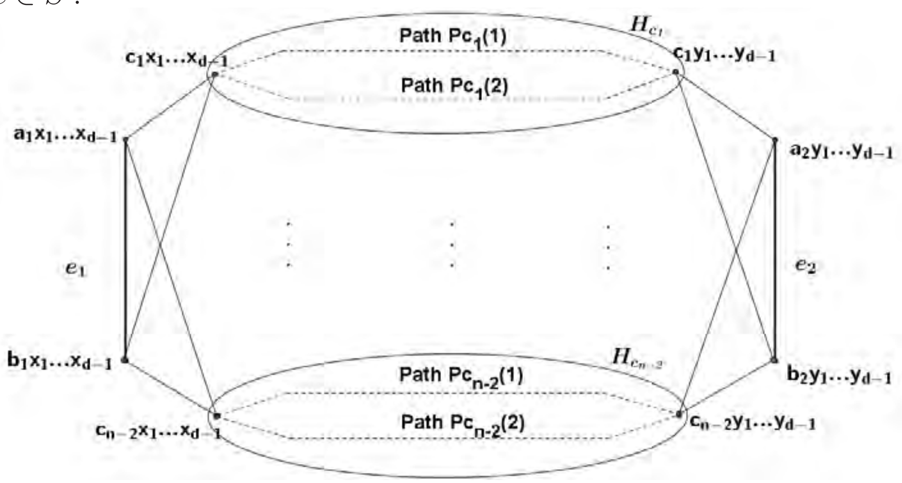


Figure 8 : $2(n-2)$ ed-paths in Q

Observe that the paths in Q are edge-disjoint and their internal vertices are not in H_{a_1} or H_{b_1}. Thus the paths in Q together with the paths in Pa_1 and Pb_1 are edge-disjoint and they induce $2(n-2) + 2(d-1)(n-1) = 2d(n-1) - 2$ id-paths from e_1 to e_2 in $L(G)$.

Case 1.2 : $|\{a_1, b_1\} \cap \{a_2, b_2\}| = 1$.

Without loss of generality, assume that $a_1 = a_2$. Then we have $(d-1)(n-1)$ paths in Pa_1.

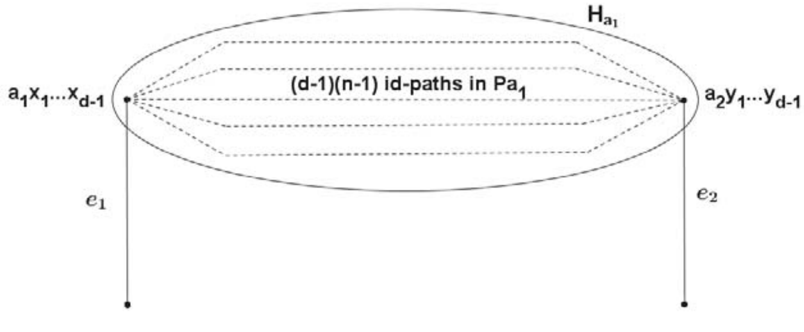


Figure 9 : ed-paths in Pa_1

Let Qb_1b_2 be the set of the following paths:

$b_1x_1...x_{d-1} - Pb_1(s_{b_1,1}) - b_1y_1...y_{d-1} \sim a_2y_1...y_{d-1}$;
$b_1x_1...x_{d-1} - Pb_1(s_{b_1,2}) - b_1y_1...y_{d-1} \sim b_2y_1...y_{d-1}$;
$a_1x_1...x_{d-1} \sim b_2x_1...x_{d-1} - Pb_2(s_{b_2,1}) - b_2y_1...y_{d-1}$;
$b_1x_1...x_{d-1} \sim b_2x_1...x_{d-1} - Pb_2(s_{b_2,2}) - b_2y_1...y_{d-1}$; and
$b_1x_1...x_{d-1} \sim s_{b_1,i} \sim s_{b_2,i} - Pb_2'(s_{b_2,i}) - b_2y_1...y_{d-1}$, $i \in \{3, ..., (d-1)(n-1)\}$.

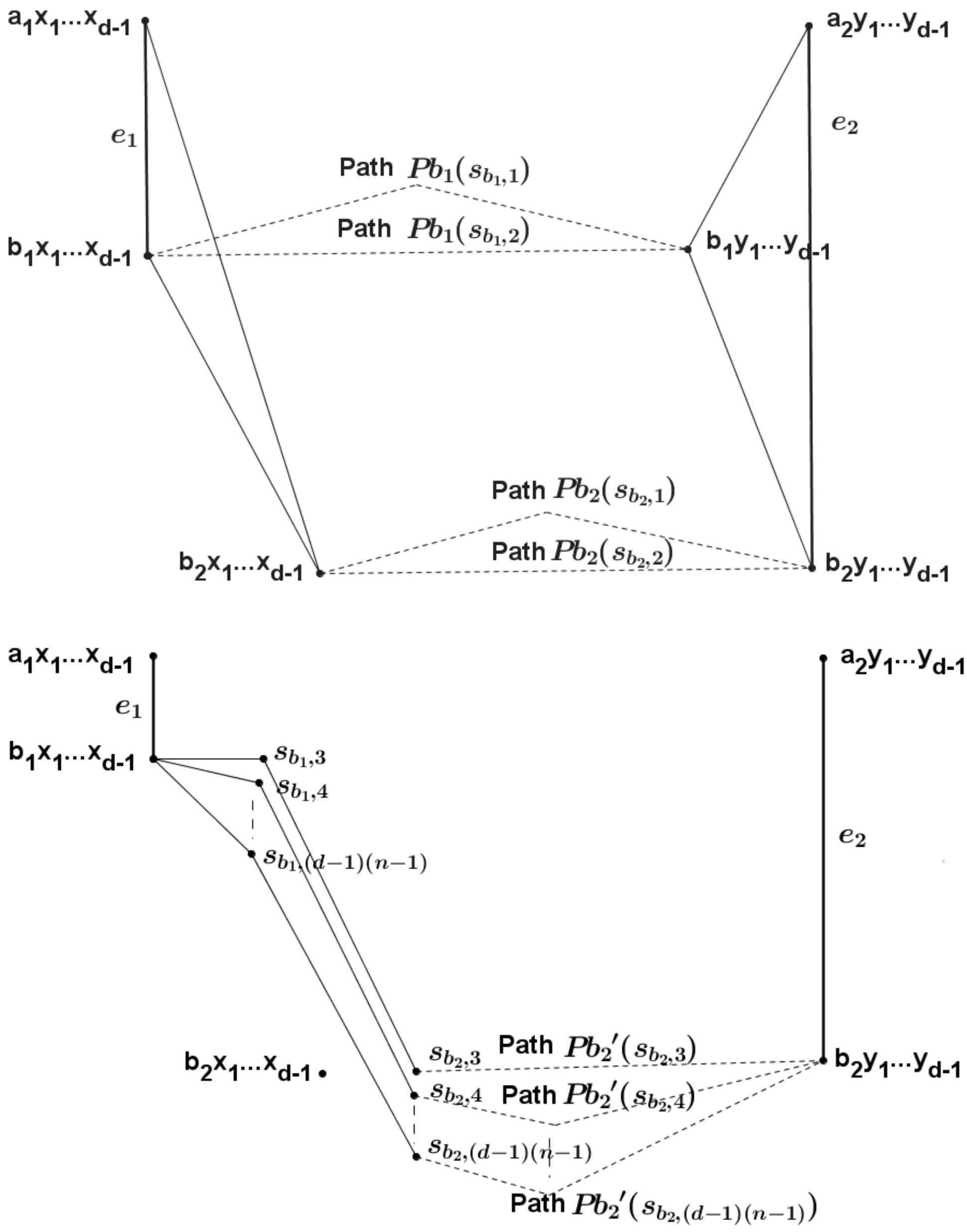


Figure 10 : ed-paths in Qb_1b_2

Observe that Qb_1b_2 has $(d-1)(n-1)+2$ paths which are edge-disjoint and their internal vertices are not in H_{a_1} but are in H_{b_1} or H_{b_2}.

Let Q be the set Q as in Case 1.1 but with $S' = S \setminus \{a_1, b_1, b_2\}$.

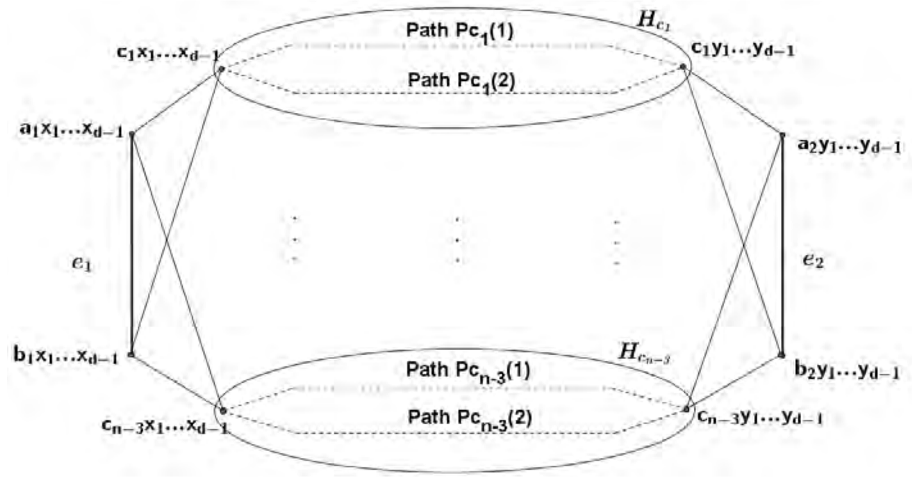


Figure 11 : $2(n-3)$ ed-paths in Q

Then Q has $2(n-3)$ paths which are edge-disjoint and their internal vertices are not in H_{a_1}, H_{b_1} or H_{b_2}. Thus the paths in Q together with the paths in Pa_1 and Qb_1b_2 are edge-disjoint and they induce $2(n-3)+(d-1)(n-1)+(d-1)(n-1)+2 = 2d(n-1)-2$ id-paths from e_1 to e_2 in $L(G)$.

Case 1.3 : $\{a_1, b_1\} \cap \{a_2, b_2\} = \phi$.

Let Qb_1b_2 be the set Qb_1b_2 as in Case 1.2 . Let Qa_1a_2 be the set of the following paths:

$a_1x_1...x_{d-1} - Pa_1(s_{a_1,1}) - a_1y_1...y_{d-1} \sim a_2y_1...y_{d-1}$;
$a_1x_1...x_{d-1} - Pa_1(s_{a_1,2}) - a_1y_1...y_{d-1} \sim b_2y_1...y_{d-1}$;
$a_1x_1...x_{d-1} \sim a_2x_1...x_{d-1} - Pa_2(s_{a_2,1}) - a_2y_1...y_{d-1}$;
$b_1x_1...x_{d-1} \sim a_2x_1...x_{d-1} - Pa_2(s_{a_2,2}) - a_2y_1...y_{d-1}$; and
$a_1x_1...x_{d-1} \sim s_{a_1,i} \sim s_{a_2,i} - Pa_2'(s_{a_2,i}) - a_2y_1...y_{d-1}$, $i \in \{3, ..., (d-1)(n-1)\}$.

Observe that Qa_1a_2 has $(d-1)(n-1)+2$ paths which are edge-disjoint and their internal vertices are not in H_{b_1} or H_{b_2} but in H_{a_1} or H_{a_2}.

Let Q be the set Q as in Case 1.1 but with $S' = S \setminus \{a_1, a_2, b_1, b_2\}$. Then Q has $2(n-4)$ paths which are edge-disjoint and their internal vertices are not in $H_{a_1}, H_{a_2}, H_{b_1}$ or H_{b_2}. Thus the paths in Q together with the paths in Qa_1a_2 and Qb_1b_2 are edge-disjoint and they induce $2(n-4)+(d-1)(n-1)+2+(d-1)(n-1)+2 = 2d(n-1)-2$ id-paths from e_1 to e_2 in $L(G)$.

Case 2 : $z_1 \neq z_2$.

Without loss of generality, assume that z_1 and z_2 are the first and second positions, respectively.

Let $a_1, a_2, b_1, b_2, c_1, c_2, x_1, ..., x_{d-2}, y_1, ..., y_{d-2}$ be the elements of S such that $a_1c_1x_1...x_{d-2}$ and $b_1c_1x_1...x_{d-2}$ are the end vertices of e_1, and $c_2a_2y_1...y_{d-2}$ and $c_2b_2y_1...y_{d-2}$ are the end vertices of e_2.

For $a, b \in S$, let H_{ab} be the induced subgraph of G whose vertices have the elements a and b at the positions z_1 and z_2, respectively. Then for $u_1, u_2, v_1, v_2 \in S$ such that $u_1 \neq u_2$ or $v_1 \neq v_2$, we have $V(H_{u_1v_1}) \cap V(H_{u_2v_2}) = \phi$. For any $a, b \in S$, H_{ab} is isomorphic to $H(d-2, n)$. Thus for any $a, b \in S$, we have $\kappa(H_{ab}) = (d-2)(n-1)$ by Theorem 2.2 and for any distinct vertices $u_1, u_2 \in H_{ab}$, there exist $(d-2)(n-1)$ id-paths from u_1 to u_2 in H_{ab} by Theorem 2.3.

For $a, b \in S$ and $u \in N_{H_{ab}}(abx_1...x_{d-2})$, let Pab be a set of $(d-2)(n-1)$ id-paths from $abx_1...x_{d-2}$ to $aby_1...y_{d-2}$ in H_{ab}, let $Pab(1), Pab(2)$ be two paths of Pab, let $Pab(u)$ be the path of Pab which contains u, and let $Pab'(u)$ be the path obtained by deleting $abx_1...x_{d-2}$ from $Pab(u)$.

Let $s_{a_1c_1,1}, s_{a_1c_1,2}, ..., s_{a_1c_1,(d-2)(n-1)}$ be the distinct vertices in $N_{H_{a_1c_1}}(a_1c_1x_1...x_{d-2})$. For any $v, w \in S$, let $s_{vw,1}, s_{vw,2}, ..., s_{vw,(d-2)(n-1)}$ be the distinct vertices in $N_{H_{vw}}(vwx_1...x_{d-2})$ such that $s_{vw,i} = rp(\ rp(s_{a_1c_1,i}, v, z_1)\ , w, z_2)$, for $i \in \{1, ..., (d-2)(n-1)\}$. Then for $i \in \{1, ..., (d-2)(n-1)\}$ and v_1, v_2, w_1 and w_2 in S such that either $v_1 = v_2$ or $w_1 = w_2$ but not both, $s_{v_1w_1,i}$ is adjacent to $s_{v_2w_2,i}$.

Case 2.1 : $c_2 \notin \{a_1, b_1\}$ and $c_1 \notin \{a_2, b_2\}$.

Let Q_1 be the set of the following paths:
$a_1c_1x_1...x_{d-2} \sim a_1a_2x_1...x_{d-2} \sim c_2a_2x_1...x_{d-2} - Pc_2a_2(s_{c_2a_2,1}) - c_2a_2y_1...y_{d-2}$;
$a_1c_1x_1...x_{d-2} - Pa_1c_1(s_{a_1c_1,1}) - a_1c_1y_1...y_{d-2} \sim a_1a_2y_1...y_{d-2} \sim c_2a_2y_1...y_{d-2}$; and
$a_1c_1x_1...x_{d-2} \sim s_{a_1c_1,i} \sim s_{a_1a_2,i} \sim s_{c_2a_2,i} - Pc_2a_2'(s_{c_2a_2,i}) - c_2a_2y_1...y_{d-2}$,
$i \in \{2, ..., (d-2)(n-1)\}$.

Then Q_1 contains $(d-2)(n-1)+1$ ed-paths whose internal vertices are in $H_{a_1a_2}$, $H_{c_2a_2}$ or $H_{a_1c_1}$.

Let Q_2 be the set of the following paths:
$b_1c_1x_1...x_{d-2} \sim b_1b_2x_1...x_{d-2} \sim c_2b_2x_1...x_{d-2} - Pc_2b_2(s_{c_2b_2,1}) - c_2b_2y_1...y_{d-2}$;
$b_1c_1x_1...x_{d-2} - Pb_1c_1(s_{b_1c_1,1}) - b_1c_1y_1...y_{d-2} \sim b_1b_2y_1...y_{d-2} \sim c_2b_2y_1...y_{d-2}$; and
$b_1c_1x_1...x_{d-2} \sim s_{b_1c_1,i} \sim s_{b_1b_2,i} \sim s_{c_2b_2,i} - Pc_2b_2'(s_{c_2b_2,i}) - c_2b_2y_1...y_{d-2}$,
$i \in \{2, ..., (d-2)(n-1)\}$.

Then Q_2 contains $(d-2)(n-1)+1$ ed-paths whose internal vertices are in $H_{b_1b_2}$, $H_{c_2b_2}$ or $H_{b_1c_1}$.

Let Q_3 be the set of the following paths:
$a_1c_1x_1...x_{d-2} \sim a_1b_2x_1...x_{d-2} - Pa_1b_2(1) - a_1b_2y_1...y_{d-2} \sim c_2b_2y_1...y_{d-2}$;
$b_1c_1x_1...x_{d-2} \sim b_1a_2x_1...x_{d-2} - Pb_1a_2(1) - b_1a_2y_1...y_{d-2} \sim c_2a_2y_1...y_{d-2}$;
$a_1c_1x_1...x_{d-2} \sim c_2c_1x_1...x_{d-2} - Pc_2c_1(1) - c_2c_1y_1...y_{d-2} \sim c_2a_2y_1...y_{d-2}$; and
$b_1c_1x_1...x_{d-2} \sim c_2c_1x_1...x_{d-2} - Pc_2c_1(2) - c_2c_1y_1...y_{d-2} \sim c_2b_2y_1...y_{d-2}$.

Then Q_3 contains 4 ed-paths whose internal vertices are in $H_{a_1b_2}$, $H_{b_1a_2}$ or $H_{c_2c_1}$.

Let $r_1, ..., r_{n-3}$ be the distinct elements of $S \setminus \{a_2, b_2, c_1\}$. Let Q_4 be the set of the following paths:
$a_1c_1x_1...x_{d-1} \sim a_1r_jx_1...x_{d-2} - Pa_1r_j(1) - a_1r_jy_1...y_{d-2} \sim c_2r_jy_1...y_{d-2} \sim c_2a_2y_1...y_{d-2}$;
and
$b_1c_1x_1...x_{d-1} \sim b_1r_jx_1...x_{d-2} - Pb_1r_j(1) - b_1r_jy_1...y_{d-2} \sim c_2r_jy_1...y_{d-2} \sim c_2b_2y_1...y_{d-2}$
where $j \in \{1, .., n-3\}$.

Then Q_4 contains $2(n-3)$ ed-paths whose internal vertices are in $H_{a_1r_j}$, $H_{b_1r_j}$ or $H_{c_2r_j}$ for $j \in \{1, .., n-3\}$.

Let $t_1, ..., t_{n-3}$ be the distinct elements of $S \setminus \{a_1, b_1, c_2\}$. Let Q_5 be the set of the following paths:
$a_1c_1x_1...x_{d-2} \sim t_jc_1x_1...x_{d-2} \sim t_ja_2x_1...x_{d-2} - Pt_ja_2(1) - t_ja_2y_1...y_{d-2} \sim c_2a_2y_1...y_{d-2}$;
and
$b_1c_1x_1...x_{d-2} \sim t_jc_1x_1...x_{d-2} \sim t_jb_2x_1...x_{d-2} - Pt_jb_2(1) - t_jb_2y_1...y_{d-2} \sim c_2b_2y_1...y_{d-2}$
where $j \in \{1, .., n-3\}$.

Then Q_5 contains $2(n-3)$ ed-paths whose internal vertices are in $H_{t_jc_1}$, $H_{t_ja_2}$ or $H_{t_jb_2}$ for $j \in \{1, .., n-3\}$.

Since the internal vertices of the paths in Q_1, Q_2, Q_3, Q_4 and Q_5 are in disjoint induced subgraphs of G, the paths in Q_1, Q_2, Q_3, Q_4 and Q_5 are edge-disjoint and they induce $(d-2)(n-1)+1+(d-2)(n-1)+1+4+2(n-3)+2(n-3) =$

$2d(n-1)-2$ id-paths from e_1 to e_2 in $L(G)$.

Case 2.2 : $c_2 \in \{a_1, b_1\}$ and $c_1 \notin \{a_2, b_2\}$.

Without loss of generality, we assume that $c_2 = a_1$.

Let Q_1 be the set of the following paths:
$a_1c_1x_1...x_{d-2} \sim a_1a_2x_1...x_{d-2} - Pa_1a_2(s_{a_1a_2,1}) - a_1a_2y_1...y_{d-2}$;
$a_1c_1x_1...x_{d-2} - Pa_1c_1(s_{a_1c_1,1}) - a_1c_1y_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$; and
$a_1c_1x_1...x_{d-2} \sim s_{a_1c_1,i} \sim s_{a_1a_2,i} - Pa_1a_2'(s_{a_1a2,i}) - a_1a_2y_1...y_{d-2}$, $i \in \{2, ..., (d-2)(n-1)\}$.

Then Q_1 contains $(d-2)(n-1)+1$ ed-paths whose internal vertices are in $H_{a_1a_2}$ or $H_{a_1c_1}$.

Let Q_2 be the set of the following paths:
$b_1c_1x_1...x_{d-2} \sim b_1b_2x_1...x_{d-2} \sim a_1b_2x_1...x_{d-2} - Pa_1b_2(s_{a_1b_2,1}) - a_1b_2y_1...y_{d-2}$;
$a_1c_1x_1...x_{d-2} \sim a_1b_2x_1...x_{d-2} - Pa_1b_2(s_{a_1b_2,2}) - a_1b_2y_1...y_{d-2}$;
$b_1c_1x_1...x_{d-2} - Pb_1c_1(s_{b_1c_1,1}) - b_1c_1y_1...y_{d-2} \sim a_1c_1y_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$;
$b_1c_1x_1...x_{d-2} - Pb_1c_1(s_{b_1c_1,2}) - b_1c_1y_1...y_{d-2} \sim b_1b_2y_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$; and
$b_1c_1x_1...x_{d-2} \sim s_{b_1c_1,j} \sim s_{b_1b_2,j} \sim s_{a_1b_2,j} - Pa_1b_2'(s_{a_1b_2,j}) - a_1b_2y_1...y_{d-2}$, $j \in \{3, ..., (d-2)(n-1)\}$.

Then Q_2 contains $(d-2)(n-1)+2$ ed-paths whose internal vertices are in $H_{b_1c_1}$, $H_{b_1b_2}$ or $H_{a_1b_2}$ but only $a_1c_1y_1...y_{d-2} \in H_{a_1c_1}$. However, the paths in Q_1 and Q_2 are edge-disjoint. Let $Q_{12} = Q_1 \cup Q_2$.

Let $t_1, ..., t_{n-2}$ be the distinct elements of $S \setminus \{a_1, b_1\}$. Let Q_3 be the set of the following paths:
$a_1c_1x_1...x_{d-2} \sim t_gc_1x_1...x_{d-2} \sim t_ga_2x_1...x_{d-2} - Pt_ga_2(1) - t_ga_2y_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$;
$b_1c_1x_1...x_{d-2} \sim t_gc_1x_1...x_{d-2} \sim t_gb_2x_1...x_{d-2} - Pt_gb_2(1) - t_gb_2y_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$;
and
$b_1c_1x_1...x_{d-2} \sim b_1a_2x_1...x_{d-2} - Pb_1a_2(1) - b_1a_2y_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$, $g \in \{1, ..., n-2\}$.

Then Q_3 contains $2(n-2)+1$ ed-paths whose internal vertices are in $H_{t_gc_1}$, $H_{t_ga_2}$, $H_{t_gb_2}$ or $H_{b_1a_2}$ for $g \in \{1, ..., n-2\}$.

Let $r_1, ..., r_{n-3}$ be the distinct elements of $S \setminus \{a_2, b_2, c_1\}$. Let Q_4 be the set of the following paths:
$a_1c_1x_1...x_{d-2} \sim a_1r_qx_1...x_{d-2} - Pa_1r_q(1) - a_1r_qy_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$; and
$b_1c_1x_1...x_{d-2} \sim b_1r_qx_1...x_{d-2} - Pb_1r_q(1) - b_1r_qy_1...y_{d-2} \sim a_1r_qy_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$, $q \in \{1, ..., n-3\}$.

Then Q_4 contains $2(n-3)$ ed-paths whose internal vertices are in $H_{a_1r_q}$ or $H_{b_1r_q}$ for $q \in \{1, ..., n-3\}$.

Since the internal vertices of the paths in Q_{12}, Q_3 and Q_4 are in disjoint induced subgraphs of G. Thus the paths in Q_{12}, Q_3 and Q_4 are edge-disjoint and they induce $(d-2)(n-1)+1+(d-2)(n-1)+2+2(n-2)+1+2(n-3) = 2d(n-1)-2$ id-paths from e_1 to e_2 in $L(G)$.

Case 2.3 : $c_2 \notin \{a_1, b_1\}$ and $c_1 \in \{a_2, b_2\}$.

Similar to Case 2.2 .

Case 2.4 : $c_2 \in \{a_1, b_1\}$ and $c_1 \in \{a_2, b_2\}$.

Without loss of generality, we assume that $a_1 = c_2$ and $a_2 = c_1$. Then Pa_1a_2 contains $(d-2)(n-1)$ paths in $H_{a_1a_2}$.

Let Q_1 be the set of the following paths:
$b_1a_2x_1...x_{d-2} \sim b_1b_2x_1...x_{d-2} - Pb_1b_2(s_{b_1b_2,1}) - b_1b_2y_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$;
$b_1a_2x_1...x_{d-2} - Pb_1a_2(s_{b_1a_2,1}) - b_1a_2y_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$;

$a_1a_2x_1...x_{d-2} \sim a_1b_2x_1...x_{d-2} - Pa_1b_2(s_{a_1b_2,1}) - a_1b_2y_1...y_{d-2}$; and
$b_1a_2x_1...x_{d-2} \sim s_{b_1a_2,i} \sim s_{b_1b_2,i} \sim s_{a_1b_2,i} - Pa_1b_2'(s_{a_1b_2,i}) - a_1b_2y_1...y_{d-2}$, $i \in \{2, ..., (d-2)(n-1)\}$.

Then Q_1 contains $1+1+1+(d-2)(n-1)-1 = (d-2)(n-1)+2$ ed-paths whose internal vertices are in $H_{b_1b_2}$, $H_{b_1a_2}$ or $H_{a_1b_2}$.

Let $r_1, ..., r_{n-2}$ be the distinct elements of $S \setminus \{a_2, b_2\}$. Let Q_2 be the set of the following paths:
$a_1a_2x_1...x_{d-2} \sim a_1r_jx_1...x_{d-2} - Pa_1r_j(1) - a_1r_jy_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$; and
$b_1a_2x_1...x_{d-2} \sim b_1r_jx_1...x_{d-2} - Pb_1r_j(1) - b_1r_jy_1...y_{d-2} \sim a_1r_jy_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$ where $j \in \{1, ..., n-2\}$.

Then Q_2 contains $2(n-2)$ ed-paths whose internal vertices are in $H_{a_1r_j}$ or $H_{b_1r_j}$ for $j \in \{1, ..., n-2\}$.

Let $t_1, ..., t_{n-2}$ be the distinct elements of $S \setminus \{a_1, b_1\}$. Let Q_3 be the set of the following paths:
$a_1a_2x_1...x_{d-2} \sim t_ja_2x_1...x_{d-2} - Pt_ja_2(1) - t_ja_2y_1...y_{d-2} \sim a_1a_2y_1...y_{d-2}$; and
$b_1a_2x_1...x_{d-2} \sim t_ja_2x_1...x_{d-2} \sim t_jb_2x_1...x_{d-2} - Pt_jb_2(1) - t_jb_2y_1...y_{d-2} \sim a_1b_2y_1...y_{d-2}$ where $j \in \{1, ..., n-2\}$.

Then Q_3 contains $2(n-2)$ ed-paths whose internal vertices are in $H_{t_ja_2}$ or $H_{t_jb_2}$ for $j \in \{1, ..., n-2\}$.

Since the internal vertices of the paths in Pa_1a_2, Q_1, Q_2 and Q_3 are in disjoint induced subgraphs of G. Thus the paths in Pa_1a_2, Q_1, Q_2 and Q_3 are edge-disjoint and they induce $(d-2)(n-1)+(d-2)(n-1)+2+2(n-2)+2(n-2) = 2d(n-1)-2$ id-paths from e_1 to e_2 in $L(G)$. □

The next theorem is the characterization of the graphs whose line graphs are Hamming graphs.

Theorem 3.6. Let G be a graph where $L(G)$ is a Hamming graph different from K_3. Then $L(G)$ is $H(1, n)$ or $H(2, n)$ for some positive integer n. Moreover, G is $K_{1,n}$ if $L(G)$ is $H(1, n)$ and G is $K_{n,n}$ if $L(G)$ is $H(2, n)$.

Proof. Let $L(G)$ be $H(d, n)$ for some positive integers d and n where $H(d, n) \neq H(1, 3)$.
Case 1: $n = 1$. $L(G)$ is the empty graph with one vertex. Hence G is $K_{1,1}$.
Case 2: $n \geq 2$. Let $S = \{1, 2, ..., n\}$ be the alphabet set of this Hamming graph.

Case 2.1: $d \geq 3$. Let $x_1, ..., x_{d-3} \in S$ then $111x_1...x_{d-3}, 211x_1...x_{d-3}, 121x_1...x_{d-3}, 112x_1...x_{d-3}$ are the vertices which induced $K_{1,3}$ in $L(G)$. By Theorem 2.9, this case does not occur.

Case 2.2: $d = 1$. $L(G)$ is a complete graph different from K_3. Hence, G is $K_{1,n}$ by Theorem 2.8.

Case 2.3: $d = 2$. Let $H = L(G)$. Observe that the induced subgraphs $H[\{11, 12, ..., 1n\}]$, $H[\{21, 22, ..., 2n\}]$,...,$H[\{n1, n2, ..., nn\}]$, $H[\{11, 21, ..., n1\}]$, $H[\{12, 22, ..., n2\}]$,...,$H[\{1n, 2n, ..., nn\}]$ are all of the cliques of $L(G)$.

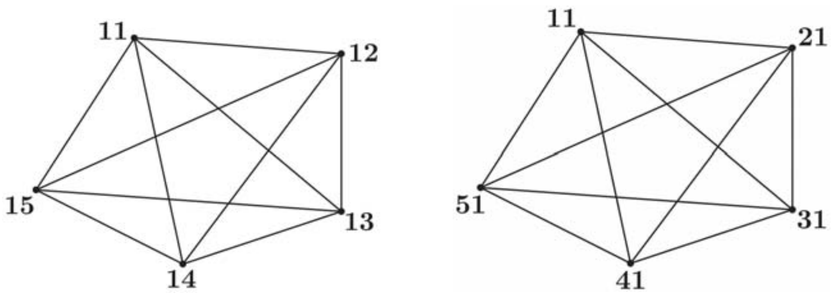


Figure 12 : Induced subgraphs $H[\{11, 12, 13, 14, 15\}]$ and $H[\{11, 21, 31, 41, 51\}]$ in $L(G)$ when $n = 5$

Moreover, each vertex of $L(G)$ is in exactly two cliques and each edge of $L(G)$ is in exactly one clique.

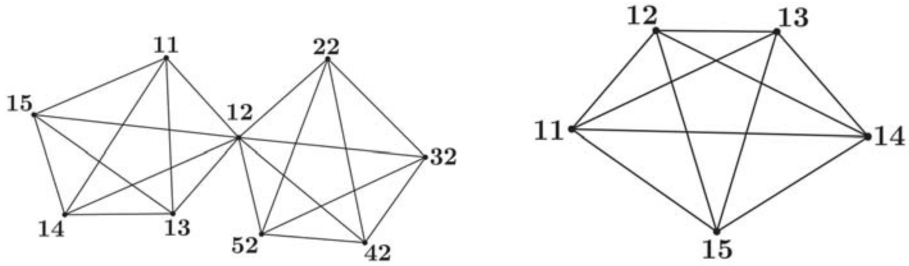


Figure 13 : The vertex 12 is in $H[\{11, 12, 13, 14, 15\}]$ and $H[\{12, 22, 32, 42, 52\}]$ and the edge from 12 to 13 is in $H[\{11, 12, 13, 14, 15\}]$ in $L(G)$ when $n = 5$

Hence these cliques form the Krauz partition of $L(G)$ and these cliques are induced by the stars at the vertices in G by Theorem 2.10. For $i \in S$, let u_i be the vertex in G whose star induce $H[\{i1, i2, ..., in\}]$ and let v_i be the vertex in G whose star induce $H[\{1i, 2i, ..., ni\}]$.

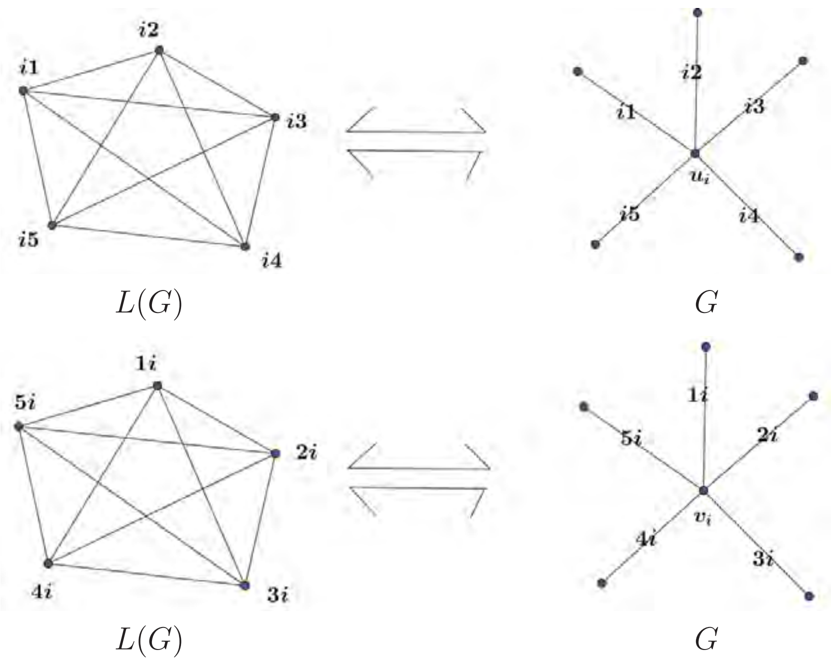
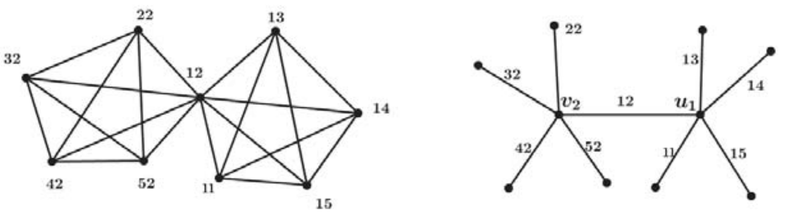


Figure 14 : The stars of u_i and v_i in G induce $H[\{i1, i2, ..., i5\}]$ and $H[\{1i, 2i, ..., 5i\}]$ in $L(G)$, respectively, when $n = 5$

Then for each $i, j \in S$, the edge ij in G (which induce the vertex ij in $L(G)$) has u_i and v_j as the end vertices.



$L(G)$ G

Figure 15 : When $n = 5$, the vertex 12 in $L(G)$ is in $H[\{11, 12, 13, 14, 15\}]$ and $H[\{12, 22, 32, 42, 52\}]$. Therefore the edge 12 in G is in the stars of u_1 and v_2. Thus, u_1 and v_2 are the end vertices of the edge 12 in G.

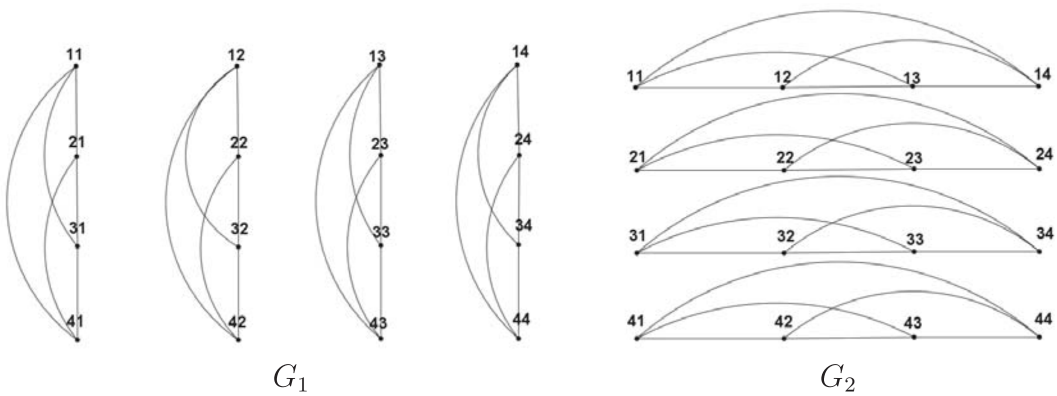
Therefore, u_i is not adjacent to u_j and v_i is not adjacent to v_j for all distinct i and j in S, but u_i are adjacent to v_j for all $i, j \in S$. Hence, G is bipartite with bipartition $(\{u_1, ..., u_n\}, \{v_1, ..., v_n\})$ and so G is $K_{n,n}$ as required. □

The last theorem for this chapter is the result on the chromatic numbers of line graphs of Hamming graphs.

Theorem 3.7. Let G be a Hamming graph $H(d, n)$ with degree $k = d(n-1)$. Then $\chi(L(G)) = \chi^1(G) = k$ if n is even; otherwise, $\chi(L(G)) = \chi^1(G) = k + 1$.

Proof. If n is odd, then $|V(G)| = n^d$ which is odd. Therefore, G has no 1-factor. That is G is not 1-factorable. Thus $\chi^1(G) = k + 1$ by Theorem 2.12 and 2.11.

Now assume that n is even. For each $i \in \{1, ..., d\}$, let G_i be the spanning subgraph of G whose vertices are adjacent when they differ exactly at position i.



G_1 G_2

Figure 16 : The spanning subgraphs G_1 and G_2 of G when G is $H(2, 4)$

Observe that G_i has n^{d-1} components and each component is a clique in G which is isomorphic to K_n. Since n is even, each component above is 1-factorable by Theorem 2.13, and, thus, G_i is 1-factorable for each $i \in \{1, ..., d\}$. Since any two cliques in G have no common edge, G_i and G_j have no common edge for all $i \neq j$. Thus, G is 1-factorable and $\chi^1(G) = k$ by Theorem 2.12. □

Chapter 4

Line graphs of Johnson graphs

In this chapter we present the results on the connectivity of line graphs of Johnson graphs. In addition we characterize the graphs whose line graphs are Johnson graphs. Since $J(n,m)$ is isomorphic to $J(n,n-m)$ [3, p 180], throughout this chapter we consider only the Johnson graphs $J(n,m)$ with $n \geq 2m$.

Note that a Johnson graphs $J(n,1)$ is the complete graph K_n, so its line graph has connectivity $2(n-1)-2$ by Theorem 3.4. Now we consider the connectivity of line graph of Johnson graphs $J(n,m)$ with $m \geq 2$. We start with the case $m=2$.

Lemma 4.1. Let G be a Johnson graph $J(n,2)$ with degree $k=2(n-2)$. Then $\kappa(L(G))=2k-2$.

Proof. Observe that $n \geq 4$. Let S be the alphabet set of this Johnson graph.

Let e_1 and e_2 be two distinct edges in G. Then e_1 and e_2 are two distinct vertices in $L(G)$. By Theorem 2.3 and Lemma 3.2, to show that $\kappa(L(G))=2k-2$ it suffices to display $2k-2$ id-paths from e_1 to e_2 in $L(G)$.

If e_1 and e_2 are adjacent, then there exist $2k-2$ id-paths from e_1 to e_2 in $L(G)$ by Lemma 3.3. We may now suppose that e_1 and e_2 are not adjacent. Since ed-paths from the end vertices of e_1 to the end vertices of e_2 in G induce e_1-e_2 id-paths in $L(G)$, to finish the proof it suffices to display $2k-2=4n-10$ ed-paths from the end vertices of e_1 to the end vertices of e_2 in G.

Let a_1, b_1, c_1, a_2, b_2 and c_2 be the elements of S such that $\{a_1,c_1\}$ and $\{b_1,c_1\}$ are the end vertices of e_1, and $\{a_2,c_2\}$ and $\{b_2,c_2\}$ are the end vertices of e_2.

Case 1 : $c_1=c_2$.

Since e_1 and e_2 are not adjacent, the elements a_1,b_1,a_2,b_2 are mutually distinct. Let $S'=S\setminus\{a_1,b_1,a_2,b_2,c_1\}$, then $|S'|=n-5$. The $4n-10$ required paths are:

$\{a_1,c_1\},\{a_2,c_1\}$;
$\{a_1,c_1\},\{b_2,c_1\}$;
$\{b_1,c_1\},\{a_2,c_1\}$;
$\{b_1,c_1\},\{b_2,c_1\}$;
$\{a_1,c_1\},\{a_1,b_2\},\{b_2,c_1\}$;
$\{a_1,c_1\},\{a_1,a_2\},\{a_2,c_1\}$;
$\{b_1,c_1\},\{b_1,a_2\},\{a_2,c_1\}$;
$\{b_1,c_1\},\{b_1,b_2\},\{b_2,c_1\}$;
$\{a_1,c_1\},\{a_1,b_1\},\{a_1,a_2\},\{a_2,b_2\},\{a_2,c_1\}$;
$\{b_1,c_1\},\{a_1,b_1\},\{b_1,b_2\},\{a_2,b_2\},\{b_2,c_1\}$;
$\{a_1,c_1\},\{x,c_1\},\{a_2,c_1\}$;
$\{b_1,c_1\},\{x,c_1\},\{b_2,c_1\}$;
$\{a_1,c_1\},\{a_1,x\},\{a_2,x\},\{a_2,c_1\}$;
$\{b_1,c_1\},\{b_1,x\},\{b_2,x\},\{b_2,c_1\}$

where $x \in S'$.

Case 2 : $c_1 \neq c_2$.

Case 2.1 : $c_1 \in \{a_2,b_2\}$ or $c_2 \in \{a_1,b_1\}$.

Without loss of generality, let $c_1=a_2$. Then $c_2 \notin \{a_1,b_1\}$ since e_1 and e_2 are not adjacent.

Case 2.1.1 : $b_2 \notin \{a_1,b_1\}$.

Then a_1,b_1,a_2,b_2 and c_2 are mutually distinct. Let $S'=S\setminus\{a_1,b_1,a_2,b_2,c_2\}$. Then $|S'|=n-5$. The $4n-10$ required paths are:

$\{a_1, a_2\}, \{a_2, c_2\}$;
$\{b_1, a_2\}, \{a_2, c_2\}$;
$\{a_1, a_2\}, \{a_1, c_2\}, \{a_2, c_2\}$;
$\{a_1, a_2\}, \{a_1, b_2\}, \{b_2, c_2\}$;
$\{a_1, a_2\}, \{a_2, b_2\}, \{a_2, c_2\}$;
$\{b_1, a_2\}, \{a_2, b_2\}, \{b_2, c_2\}$;
$\{a_1, a_2\}, \{a_1, b_1\}, \{a_1, c_2\}, \{b_2, c_2\}$;
$\{b_1, a_2\}, \{a_1, b_1\}, \{b_1, c_2\}, \{a_2, c_2\}$;
$\{b_1, a_2\}, \{b_1, c_2\}, \{b_2, c_2\}$;
$\{b_1, a_2\}, \{b_1, b_2\}, \{b_2, c_2\}$;
$\{a_1, a_2\}, \{a_2, x\}, \{a_2, c_2\}$;
$\{a_1, a_2\}, \{a_1, x\}, \{c_2, x\}, \{a_2, c_2\}$;
$\{b_1, a_2\}, \{a_2, x\}, \{c_2, x\}, \{b_2, c_2\}$;
$\{b_1, a_2\}, \{b_1, x\}, \{b_2, x\}, \{b_2, c_2\}$

where $x \in S'$.

Case 2.1.2 : $b_2 \in \{a_1, b_1\}$.

Without loss of generality, let $b_2 = a_1$. Then a_1, b_1, a_2 and c_2 are mutually distinct. Let $S' = S \setminus \{a_1, b_1, a_2, c_2\}$. Then $|S'| = n - 4$. The $4n - 10$ required paths are:

$\{a_1, a_2\}, \{a_1, c_2\}$;
$\{a_1, a_2\}, \{a_2, c_2\}$;
$\{b_1, a_2\}, \{a_2, c_2\}$;
$\{a_1, a_2\}, \{a_1, b_1\}, \{a_1, c_2\}$;
$\{b_1, a_2\}, \{b_1, c_2\}, \{a_2, c_2\}$;
$\{b_1, a_2\}, \{a_1, b_1\}, \{b_1, c_2\}, \{a_1, c_2\}$;
$\{a_1, a_2\}, \{a_1, x\}, \{a_1, c_2\}$;
$\{b_1, a_2\}, \{a_2, x\}, \{a_2, c_2\}$;
$\{a_1, a_2\}, \{a_2, x\}, \{c_2, x\}, \{a_1, c_2\}$;
$\{b_1, a_2\}, \{b_1, x\}, \{c_2, x\}, \{a_2, c_2\}$

where $x \in S'$.

Case 2.2 : $c_1 \notin \{a_2, b_2\}$ and $c_2 \notin \{a_1, b_1\}$.

Case 2.2.1 : $\{a_1, b_1\} = \{a_2, b_2\}$.

Without loss of generality, let $a_1 = a_2$ and $b_1 = b_2$. Then a_1, b_1, c_1 and c_2 are mutually distinct. Let $S' = S \setminus \{a_1, b_1, c_1, c_2\}$. Then $|S'| = n - 4$. The $4n - 10$ required paths are:

$\{a_1, c_1\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{b_1, c_2\}$;
$\{a_1, c_1\}, \{c_1, c_2\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{c_1, c_2\}, \{b_1, c_2\}$;
$\{a_1, c_1\}, \{a_1, b_1\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{a_1, b_1\}, \{b_1, c_2\}$;
$\{a_1, c_1\}, \{a_1, x\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{b_1, x\}, \{b_1, c_2\}$;
$\{a_1, c_1\}, \{c_1, x\}, \{c_2, x\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{c_1, x\}, \{c_1, c_2\}, \{c_2, x\}, \{b_1, c_2\}$

where $x \in S'$.

Case 2.2.2 : $|\{a_1, b_1\} \cap \{a_2, b_2\}| = 1$.

Without loss of generality, let $a_1 = a_2$. Then a_1, b_1, b_2, c_1 and c_2 are mutually distinct. Let $S' = S \setminus \{a_1, b_1, b_2, c_1, c_2\}$. Then $|S'| = n - 5$. The $4n - 10$ required paths are:

$\{a_1, c_1\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{b_2, c_2\}$;
$\{a_1, c_1\}, \{a_1, b_2\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{b_1, c_2\}, \{b_2, c_2\}$;
$\{b_1, c_1\}, \{c_1, b_2\}, \{b_2, c_2\}$;
$\{a_1, c_1\}, \{a_1, b_1\}, \{a_1, c_2\}$;
$\{a_1, c_1\}, \{c_1, c_2\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{c_1, c_2\}, \{b_2, c_2\}$;
$\{a_1, c_1\}, \{c_1, b_2\}, \{a_1, b_2\}, \{b_2, c_2\}$;
$\{b_1, c_1\}, \{a_1, b_1\}, \{b_1, c_2\}, \{a_1, c_2\}$;
$\{a_1, c_1\}, \{a_1, x\}, \{a_1, c_2\}$;
$\{b_1, c_1\}, \{b_1, x\}, \{b_2, x\}, \{b_2, c_2\}$;
$\{a_1, c_1\}, \{c_1, x\}, \{c_2, x\}, \{b_2, c_2\}$;
$\{b_1, c_1\}, \{c_1, x\}, \{c_1, c_2\}, \{c_2, x\}, \{a_1, c_2\}$

where $x \in S'$.

Case 2.2.3 : $\{a_1, b_1\} \cap \{a_2, b_2\} = \phi$.

Then a_1, b_1, a_2, b_2, c_1 and c_2 are mutually distinct. Let $S' = S \setminus \{a_1, b_1, a_2, b_2, c_1, c_2\}$. Then $|S'| = n - 6$. The $4n - 10$ required paths are:

$\{a_1,c_1\},\{a_1,a_2\},\{a_2,c_2\}$; $\{b_1,c_1\},\{c_1,b_2\},\{b_2,c_2\}$;
$\{a_1,c_1\},\{c_1,a_2\},\{a_2,c_2\}$; $\{b_1,c_1\},\{c_1,c_2\},\{b_2,c_2\}$;
$\{a_1,c_1\},\{a_1,b_2\},\{b_2,c_2\}$; $\{b_1,c_1\},\{b_1,c_2\},\{a_2,c_2\}$;
$\{a_1,c_1\},\{c_1,c_2\},\{a_2,c_2\}$; $\{b_1,c_1\},\{c_1,a_2\},\{a_2,b_2\},\{b_2,c_2\}$;
$\{a_1,c_1\},\{a_1,c_2\},\{a_2,c_2\}$; $\{b_1,c_1\},\{a_1,b_1\},\{b_1,c_2\},\{b_2,c_2\}$;
$\{a_1,c_1\},\{c_1,b_2\},\{a_2,b_2\},\{a_2,c_2\}$; $\{a_1,c_1\},\{a_1,x\},\{a_2,x\},\{a_2,c_2\}$;
$\{a_1,c_1\},\{a_1,b_1\},\{a_1,c_2\},\{b_2,c_2\}$; $\{b_1,c_1\},\{b_1,x\},\{b_2,x\},\{b_2,c_2\}$;
$\{b_1,c_1\},\{b_1,a_2\},\{a_2,c_2\}$; $\{a_1,c_1\},\{c_1,x\},\{c_2,x\},\{a_2,c_2\}$;
$\{b_1,c_1\},\{b_1,b_2\},\{b_2,c_2\}$; $\{b_1,c_1\},\{c_1,x\},\{c_1,c_2\},\{c_2,x\},\{b_2,c_2\}$

where $x \in S'$. □

Lemma 4.2. Let G be a Johnson graph $J(n,m)$ with degree $k = m(n-m)$ where $m \geq 3$. Then $\kappa(L(G)) = 2k-2$.

Proof. We will induct on $n+m$. By Proposition 3.4, Lemma 3.3 and Lemma 4.1, we obtain the result for Johnson graphs $J(n,m)$ with $n+m \in \{3,4,5,6,7,8\}$. Now suppose $n+m \geq 9$. Since $n \geq 2m$, we have $n-m \geq 3$.

Let S be the alphabet set of this Johnson graph. Let e_1 and e_2 be two distinct edges in G. Then e_1 and e_2 are two distinct vertices in $L(G)$. By Theorem 2.3 and Lemma 3.2, to show that $\kappa(L(G)) = 2k-2$ it suffices to display $2k-2$ id-paths from e_1 to e_2 in $L(G)$.

If e_1 and e_2 are adjacent, then there exist $2k-2$ id-paths from e_1 to e_2 in $L(G)$ by Lemma 3.3. We may now suppose that e_1 and e_2 are not adjacent.

Observe that the degree of e_1 and e_2 in $L(G)$ is $2k-2$ which equals the number of e_1-e_2 id-paths in $L(G)$ which we have to find. Therefore, to finish the proof it suffices to display disjoint paths from all neighbors of e_1 to all neighbors of e_2 in $L(G)$.

For distinct $a_1,...,a_h,b_1,...,b_p \in S$ with $h \leq m$, let $Aa_1...a_h\bar{b}_1...\bar{b}_p$ be the induced subgraphs of $J(n,m)$ whose vertices contain $a_1,...,a_h$ but do not contain $b_1,...,b_p$. Then we have $Aa_1...a_h\bar{b}_1...\bar{b}_p \cong J(n-h-p, m-h)$. By the inductive hypothesis, we have $\kappa(L(Aa_1...a_h\bar{b}_1...\bar{b}_p)) = 2(m-h)(n-m-p)-2$.

Let $a,b,c,d,x_0,...,x_{m-2},y_0,...,y_{m-2}$ be the elements of S such that $\{a,x_0,...,x_{m-2}\}$ and $\{b,x_0,...,x_{m-2}\}$ are the end vertices of e_1 and $\{c,y_0,...,y_{m-2}\}$ and $\{d,y_0,...,y_{m-2}\}$ are the end vertices of e_2. Let $z_0,...,z_{n-m-2}$ be the distinct elements in $S\setminus\{a,b,x_0,...,x_{m-2}\}$ and Let $w_0,...,w_{n-m-2}$ be the distinct elements in $S\setminus\{c,d,y_0,...,y_{m-2}\}$.

Case 1 : $\{a,b\}\cap\{c,d\} \neq \phi$.

Without loss of generality, assume that $a = c$. In the table below we define some subsets $Ea, E\bar{a}, Eab, E\bar{a}\bar{b}, Ea2, E\bar{a}2, Fa, F\bar{a}, Fab, F\bar{a}\bar{b}, Fa2$ and $F\bar{a}2$ of $V(L(G))$ by specifying their members:

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
Ea	$\{a,x_0,...,x_{m-2}\}$	$\{a,x_0,...,x_{m-2},z_j\}\setminus\{x_i\}$ and $\{a,b,x_0,...,x_{m-2}\}\setminus\{x_i\}$
$E\bar{a}$	$\{b,x_0,...,x_{m-2}\}$	$\{b,x_0,...,x_{m-2},z_j\}\setminus\{x_i\}$ and $\{x_0,...,x_{m-2},z_j\}$
Eab	$\{b,x_0,...,x_{m-2}\}$	$\{a,b,x_0,...,x_{m-2}\}\setminus\{x_i\}$
$E\bar{a}\bar{b}$	$\{a,x_0,...,x_{m-2}\}$	$\{x_0,...,x_{m-2},z_j\}$

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
$Ea2$	$\{a, b, x_0, ..., x_{m-2}\} \setminus \{x_i\}$	$\{a, x_0, ..., x_{m-2}, z_0\} \setminus \{x_i\}$
$E\bar{a}2$	$\{x_0, ..., x_{m-2}, z_j\}$	$\{b, x_0, ..., x_{m-3}, z_j\}$
Fa	$\{a, y_0, ..., y_{m-2}\}$	$\{a, y_0, ..., y_{m-2}, w_j\} \setminus \{y_i\}$ and $\{a, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$
$F\bar{a}$	$\{d, y_0, ..., y_{m-2}\}$	$\{d, y_0, ..., y_{m-2}, w_j\} \setminus \{y_i\}$ and $\{y_0, ..., y_{m-2}, w_j\}$
Fab	$\{d, y_0, ..., y_{m-2}\}$	$\{a, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$
$F\bar{a}\bar{b}$	$\{a, y_0, ..., y_{m-2}\}$	$\{y_0, ..., y_{m-2}, w_j\}$
$Fa2$	$\{a, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$	$\{a, y_0, ..., y_{m-2}, w_0\} \setminus \{y_i\}$
$F\bar{a}2$	$\{y_0, ..., y_{m-2}, w_j\}$	$\{d, y_0, ..., y_{m-3}, w_j\}$

where $i \in \{0, 1, ..., m-2\}$ and $j \in \{0, 1, ..., n-m-2\}$.

Table 1 : The sets $Ea, E\bar{a}, Eab, E\bar{a}\bar{b}$, $Ea2, E\bar{a}2, Fa, F\bar{a}, Fab, F\bar{a}\bar{b}, Fa2$ and $F\bar{a}2$

The neighbors of e_1 in $L(G)$ are the vertices in $Ea \cup E\bar{a} \cup Eab \cup E\bar{a}\bar{b}$ and the neighbors of e_2 in $L(G)$ are the vertices in $Fa \cup F\bar{a} \cup Fab \cup F\bar{a}\bar{b}$.

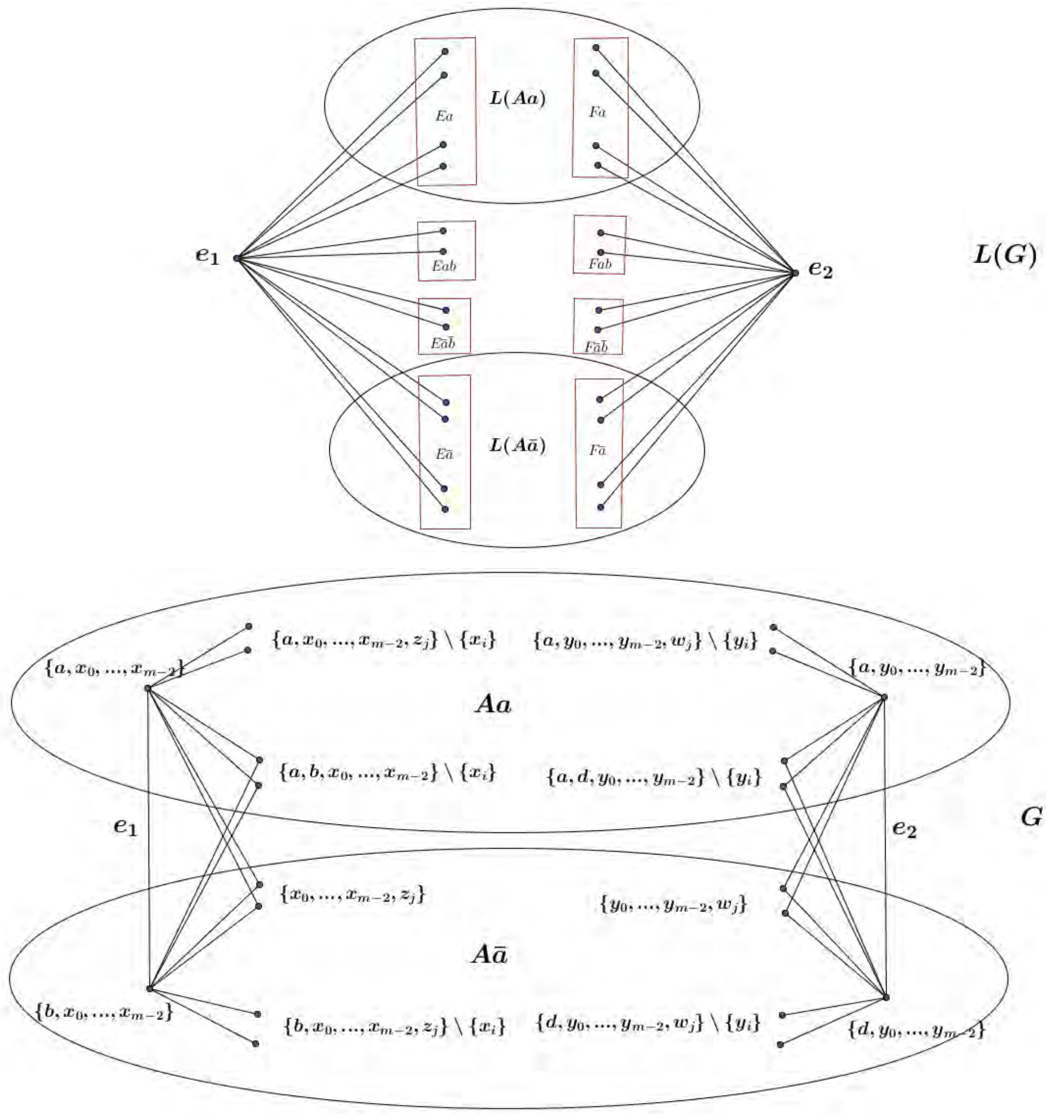


Figure 17 : The edges in G which induce the neighbors of e_1 and e_2 in $L(G)$

To get disjoint paths between all neighbors of e_1 and all neighbors of e_2 in

$L(G)$, we will display a set of disjoint paths and extend them, if necessary, via a certain matching to get the required paths.

By construction, there is a matching in $L(G)$ that matches the vertices in $Eab, E\bar{a}\bar{b}, Fab$ and $F\bar{a}\bar{b}$ to the vertices in $Ea2, E\bar{a}2, Fa2$ and $F\bar{a}2$, respectively. Let $P = (Eab \cup E\bar{a}\bar{b}) \cap (Fab \cup F\bar{a}\bar{b})$.

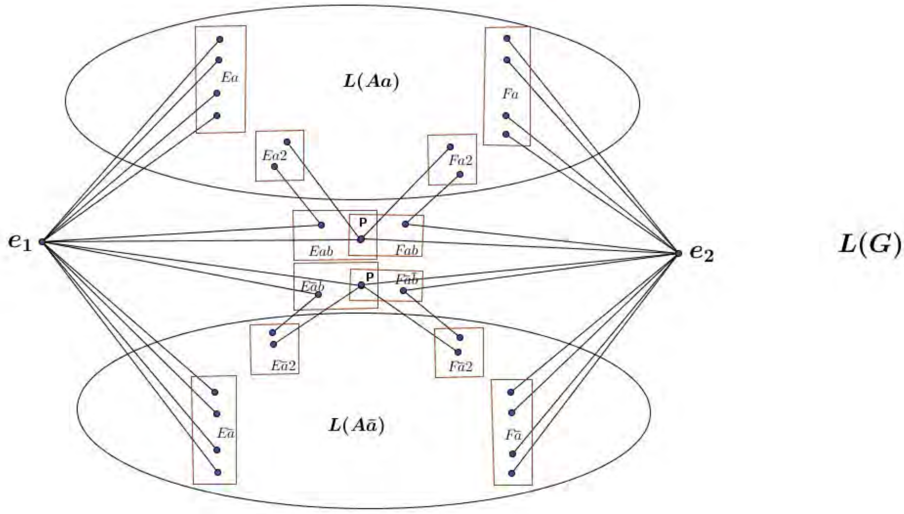


Figure 18 : The vertices in P

If P is not empty, then the vertices in P are common neighbors of e_1 and e_2 in $L(G)$, and so we can view P as a set of trivial paths from neighbors of e_1 to neighbors of e_2. Let Ea^- be the vertices in $Ea2 \cup E\bar{a}2$ which are matched to the vertices in P and let Fa^- be the vertices in $Fa2 \cup F\bar{a}2$ which are matched to the vertices in P. We now display disjoint paths from all the vertices in $(Ea \cup Ea2 \cup E\bar{a} \cup E\bar{a}2) \setminus Ea^-$ to all the vertices in $(Fa \cup Fa2 \cup F\bar{a} \cup F\bar{a}2) \setminus Fa^-$. Note that the vertices in Ea, $Ea2$, Fa and $Fa2$ are in $L(Aa)$ and the vertices in $E\bar{a}$, $E\bar{a}2$, $F\bar{a}$ and $F\bar{a}2$ are in $L(A\bar{a})$.

Since $|(Ea \cup Ea2) \setminus Ea^-| = |(Fa \cup Fa2) \setminus Fa^-| \leq (m-1)(n-m-1) + 2(m-1) = (m-1)(n-m+1) = (m-1)(n-m) + (m-1) \leq (m-1)(n-m) + 3(m-1) - 2 \leq (m-1)(n-m) + (m-1)(n-m) - 2 = 2(m-1)(n-m) - 2 = \kappa(L(Aa))$, by Theorem 2.5, there exist disjoint paths in $L(Aa)$ from all vertices in $(Ea \cup Ea2) \setminus Ea^-$ to all vertices in $(Fa \cup Fa2) \setminus Fa^-$. Let Pa be the set of these disjoint paths.

Since $|(E\bar{a} \cup E\bar{a}2) \setminus Ea^-| = |(F\bar{a} \cup F\bar{a}2) \setminus Fa^-| \leq (m-1)(n-m-1) + 2(n-m-1) = m(n-m-1) + (n-m-1) \leq m(n-m-1) + 3(n-m-1) - 2 \leq m(n-m-1) + m(n-m-1) - 2 = 2m(n-m-1) - 2 = \kappa(L(A\bar{a}))$. By Theorem 2.5, there exist disjoint paths in $L(A\bar{a})$ from all vertices in $(E\bar{a} \cup E\bar{a}2) \setminus Ea^-$ to all vertices in $(F\bar{a} \cup F\bar{a}2) \setminus Fa^-$. Let $P\bar{a}$ be the set of these disjoint paths.

Therefore, the paths in $Pa \cup P\bar{a}$ are paths from the vertices in $(Ea \cup Ea2 \cup E\bar{a} \cup E\bar{a}2) \setminus Ea^-$ to the vertices in $(Fa \cup Fa2 \cup F\bar{a} \cup F\bar{a}2) \setminus Fa^-$. Among these paths, only the vertices in $(Ea2 \cup E\bar{a}2) \setminus Ea^-$ are the end of the paths that do not begin at neighbors of e_1, and only the vertices in $(Fa2 \cup F\bar{a}2) \setminus Fa^-$ are the end of the paths that do not end at neighbors of e_2. Extend the paths via the above matching to get paths from neighbors of e_1 to neighbors of e_2. The paths in $Pa \cup P\bar{a}$ with the extension together with the paths in P are paths from all neighbors of e_1 to all neighbors of e_2 in $L(G)$.

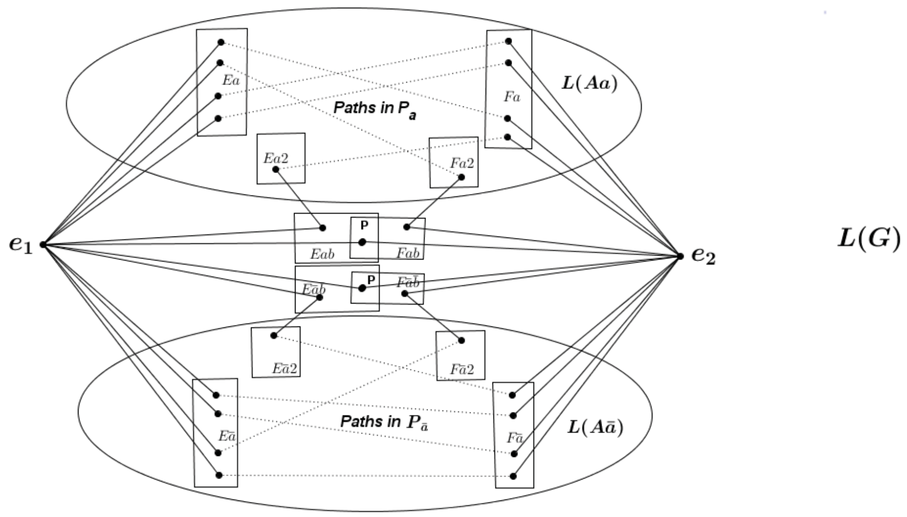


Figure 19 : The paths from all neighbors of e_1 to all neighbors of e_2 in $L(G)$

Finally we show that the above paths are disjoint. By construction and the definition of P, the vertices extended via the matching (the vertices in Eab, $E\bar{a}\bar{b}$, Fab, and $F\bar{a}\bar{b}$ which are not in P) are all distinct and they are disjoint from the vertices in P. Moreover, the extended vertices and the vertices in P are in neither $L(Aa)$ nor $L(A\bar{a})$, but the disjoint paths in Pa and $P\bar{a}$ are in disjoint induced subgraphs $L(Aa)$ and $L(A\bar{a})$, respectively. Therefore the specified paths from all neighbors of e_1 to all neighbors of e_2 in $L(G)$ are disjoint as required.

Case 2 : $\{a, b\} \cap \{c, d\} = \phi$ but $\{x_0, ..., x_{m-2}\} \cap \{y_0, ..., y_{m-2}\} \neq \phi$.

Without loss of generality, assume that $x_{m-2} = y_{m-2}$. Then we have that e_1 and e_2 are in Ax_{m-2}. By the inductive hypothesis and Theorem 2.3, there exist $2(m-1)(n-m) - 2 = \kappa(L(Ax_{m-2}))$ id-paths from e_1 to e_2 in $L(Ax_{m-2})$.

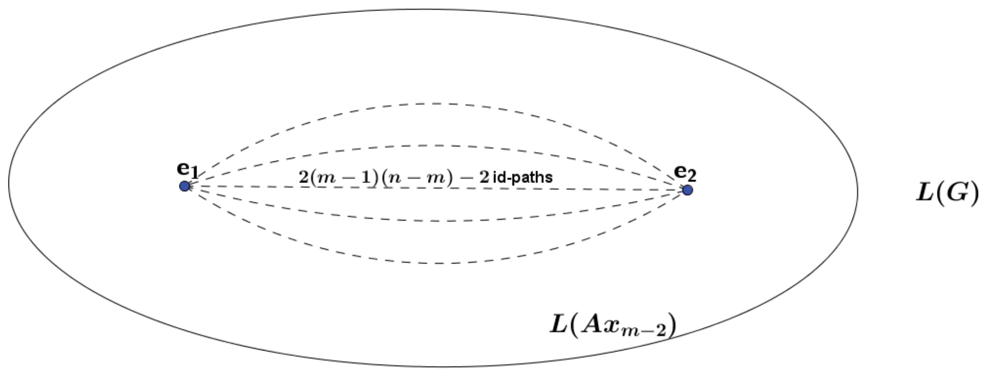


Figure 20 : $2(m-1)(n-m) - 2$ id-paths in $L(Ax_{m-2})$ from e_1 to e_2 in $L(G)$

Note that these paths contain all neighbors of e_1 and e_2 in $L(Ax_{m-2})$. To finish this case, we will display disjoint paths between all neighbors of e_1 and all neighbors of e_2 in $L(G) \setminus L(Ax_{m-2})$. To do this, we will display a set of disjoint paths in $L(A\bar{x}_{m-2})$ and extend them via a certain matching to get the required paths.

In the table below we define some subsets Exa, $Exab$, Exb, $E\bar{x}2$, $E\bar{x}3$, $E\bar{x}4$, Fxa, $Fxab$, Fxb, $F\bar{x}2$, $F\bar{x}3$ and $F\bar{x}4$ of $V(L(G))$ by specifying their members:

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
Exa	$\{a, x_0, ..., x_{m-2}\}$	$\{a, x_0, ..., x_{m-3}, z_j\}$
$Exab$	$\{a, x_0, ..., x_{m-2}\}$ and $\{b, x_0, ..., x_{m-2}\}$	$\{a, b, x_0, ..., x_{m-3}\}$
Exb	$\{b, x_0, ..., x_{m-2}\}$	$\{b, x_0, ..., x_{m-3}, z_j\}$
$E\bar{x}2$	$\{a, x_0, ..., x_{m-3}, z_j\}$	$\{b, x_0, ..., x_{m-3}, z_j\}$
$E\bar{x}3$	$\{a, b, x_0, ..., x_{m-3}\}$	$\{a, x_0, ..., x_{m-3}, z_g\}$; $g \in \{0, 1\}$
$E\bar{x}4$	$\{b, x_0, ..., x_{m-3}, z_j\}$	$\{a, b, x_0, ..., x_{m-3}\}$
Fxa	$\{c, y_0, ..., y_{m-2}\}$	$\{c, y_0, ..., y_{m-3}, w_j\}$
$Fxab$	$\{c, y_0, ..., y_{m-2}\}$ and $\{d, y_0, ..., y_{m-2}\}$	$\{c, d, y_0, ..., y_{m-3}\}$
Fxb	$\{d, y_0, ..., y_{m-2}\}$	$\{d, y_0, ..., y_{m-3}, w_j\}$
$F\bar{x}2$	$\{c, y_0, ..., y_{m-3}, w_j\}$	$\{d, y_0, ..., y_{m-3}, w_j\}$
$F\bar{x}3$	$\{c, d, y_0, ..., y_{m-3}\}$	$\{c, y_0, ..., y_{m-3}, w_g\}$; $g \in \{0, 1\}$
$F\bar{x}4$	$\{d, y_0, ..., y_{m-3}, w_j\}$	$\{c, d, y_0, ..., y_{m-3}\}$

where $i \in \{0, 1, ..., m-2\}$ and $j \in \{0, 1, ..., n-m-2\}$.

Table 2 : The sets $Exa, Exab, Exb, E\bar{x}2, E\bar{x}3, E\bar{x}4$, $Fxa, Fxab, Fxb, F\bar{x}2$, $F\bar{x}3$ and $F\bar{x}4$

The neighbors of e_1 in $L(G)\backslash L(Ax_{m-2})$ are the vertices in $Exa \cup Exab \cup Exb$ and the neighbors of e_2 in $L(G) \setminus L(Ax_{m-2})$ are the vertices in $Fxa \cup Fxab \cup Fxb$.

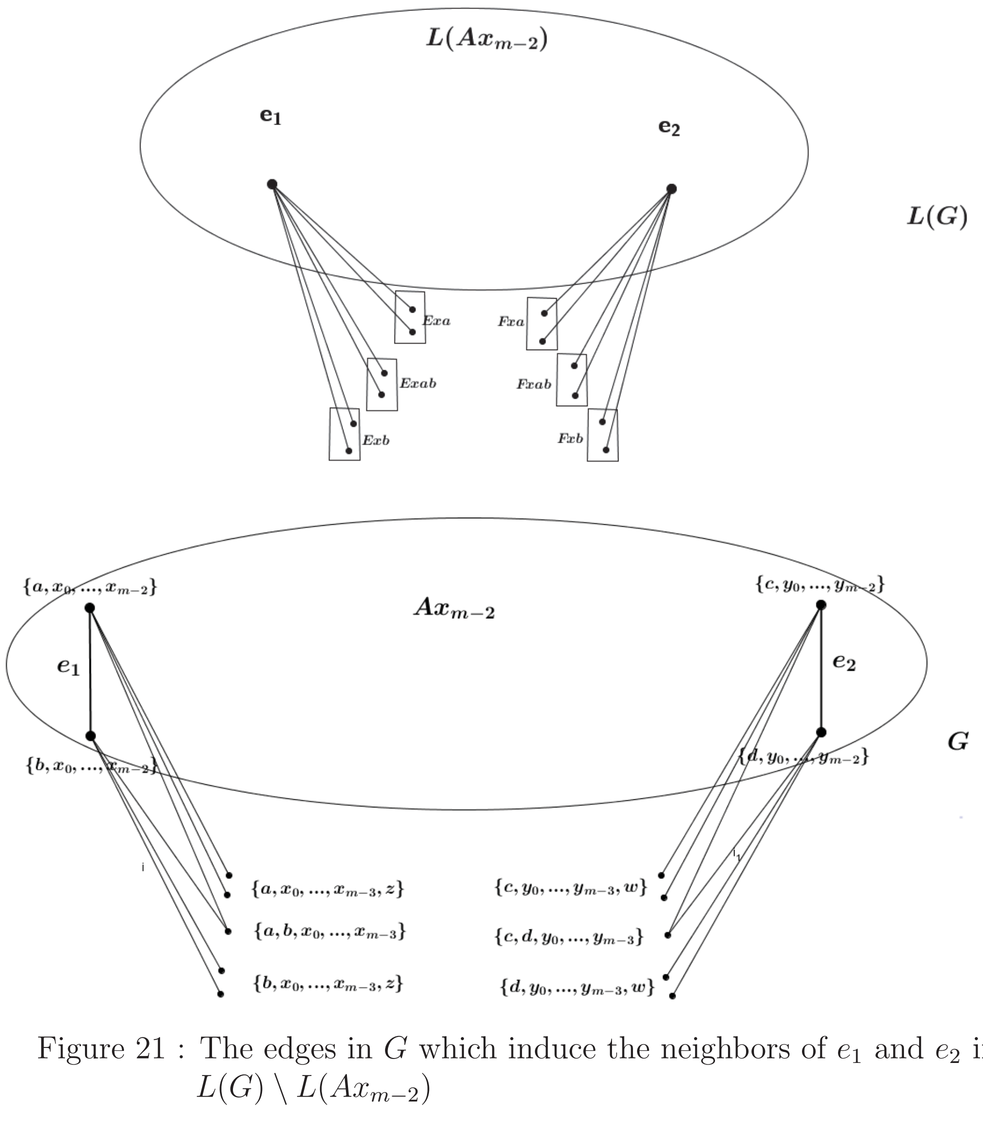


Figure 21 : The edges in G which induce the neighbors of e_1 and e_2 in $L(G) \setminus L(Ax_{m-2})$

By construction, there is a matching in $L(G)$ that matches the vertices in $Exa, Exab, Exb, Fxa, Fxab$ and Fxb to the vertices in $E\bar{x}2, E\bar{x}3, E\bar{x}4, F\bar{x}2, F\bar{x}3$ and $F\bar{x}4$, respectively.

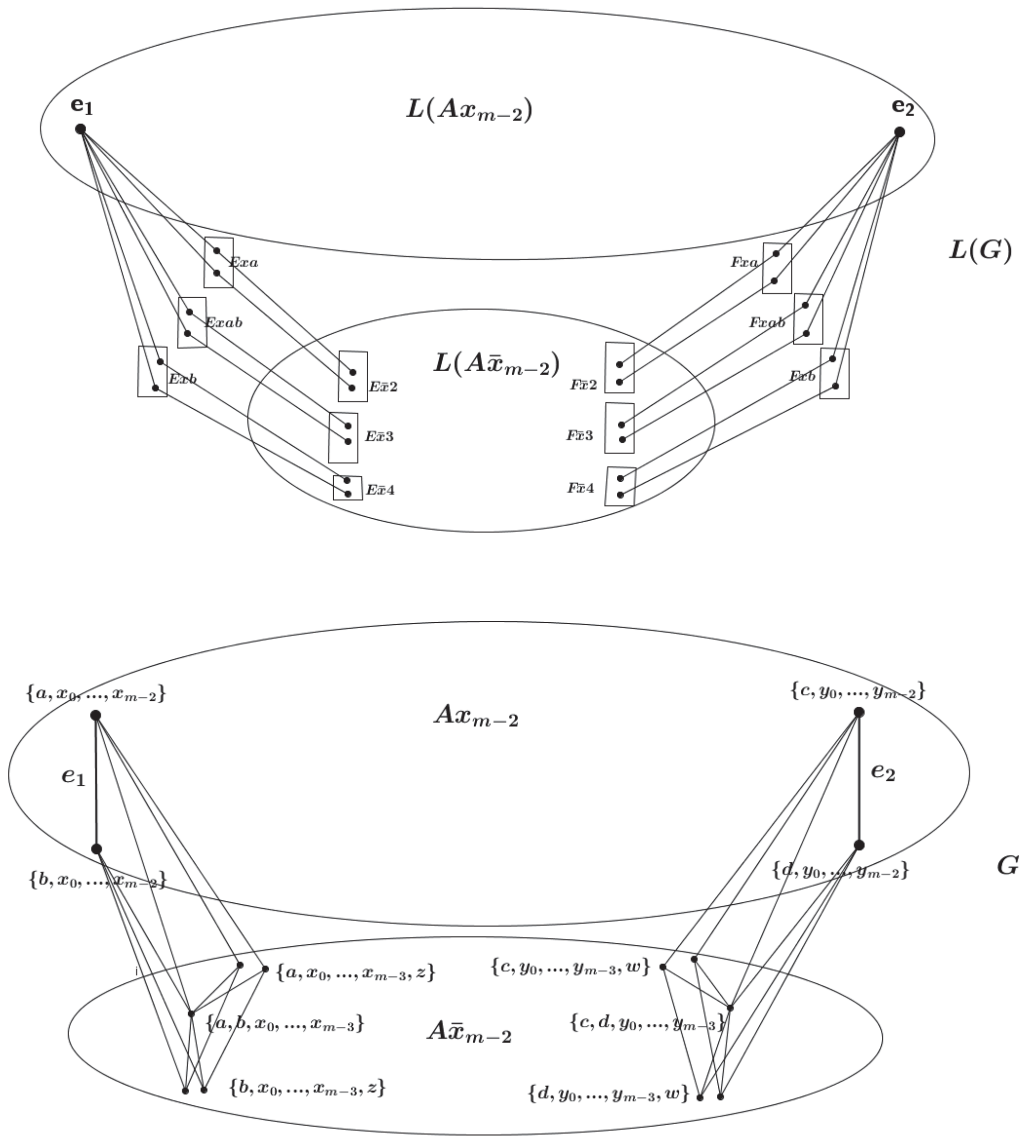


Figure 22 : The edges in G which induce the vertices in $E\bar{x}2, E\bar{x}3, E\bar{x}4, F\bar{x}2, F\bar{x}3$ and $F\bar{x}4$

Let $Ex_T = E\bar{x}2 \cup E\bar{x}3 \cup E\bar{x}4$ and $Fx_T = F\bar{x}2 \cup F\bar{x}3 \cup F\bar{x}4$.

We now display disjoint paths in $L(A\bar{x}_{m-2})$ from all vertices in Ex_T to all vertices in Fx_T. Note that the vertices in Ex_T and Fx_T are in $L(A\bar{x}_{m-2})$. Since $|Ex_T| = |Fx_T| = 2(n-m) \leq 4(n-m-1)-2 \leq 2m(n-m-1)-2 = \kappa(L(A\bar{x}_{m-2}))$, by Theorem 2.5, there exist $2(n-m)$ disjoint paths in $L(A\bar{x}_{m-2})$ from all vertices in Ex_T to all vertices in Fx_T. Let $P\bar{x}$ be the set of these paths.

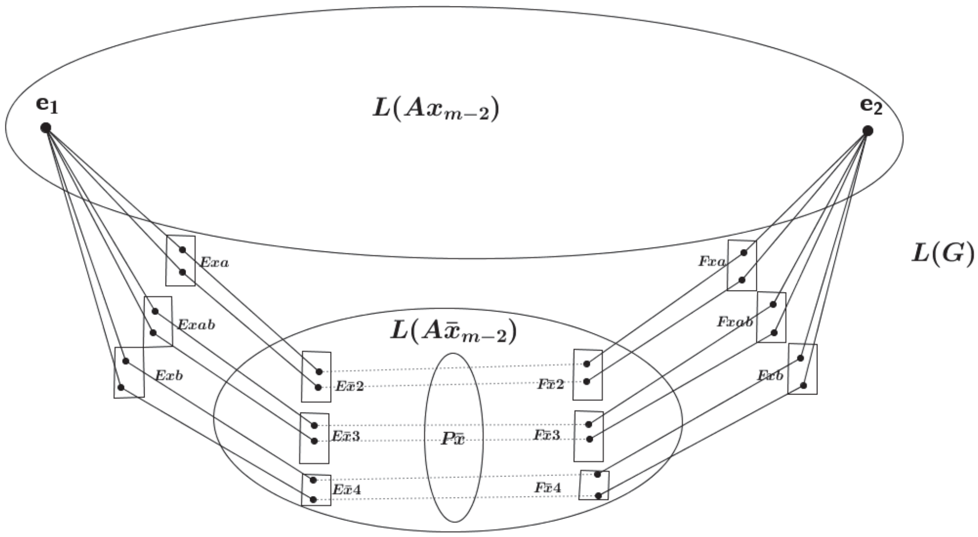


Figure 23 : Disjoint paths in $P\bar{x}$

Extend the paths in $P\bar{x}$ via the above matching to get the paths from all neighbors of e_1 to all neighbors of e_2 in $L(G) \setminus L(Ax_{m-2})$. Finally we show that the extended paths are disjoint.

By construction the vertices extended via the matching (all neighbors of e_1 and e_2 in $L(G) \setminus L(Ax_{m-2})$) are distinct and are in $L(G) \setminus L(Ax_{m-2})$ but not in $L(A\bar{x}_{m-2})$, but the disjoint paths in $P\bar{x}$ are in $L(A\bar{x}_{m-2})$. Therefore the extended paths from all neighbors of e_1 to all neighbors of e_2 in $L(G)\setminus L(Ax_{m-2})$ are disjoint as required.

Case 3 : $\{a, b\} \cap \{c, d\} = \phi$ and $\{x_0, ..., x_{m-2}\} \cap \{y_0, ..., y_{m-2}\} = \phi$.

Case 3.1 : $\{a, b, c, d, x_0, ..., x_{m-2}, y_0, ..., y_{m-2}\} \neq S$.

Let $z \in S \setminus \{a, b, c, d, x_0, ..., x_{m-2}, y_0, ..., y_{m-2}\}$. Then the edges e_1 and e_2 are in $A\bar{z}$. By the inductive hypothesis and Theorem 2.3, there exist $2m(n - m - 1) - 2 = \kappa(L(A\bar{z}))$ id-paths from e_1 to e_2 in $L(A\bar{z})$. Note that these paths contain all the neighbors of e_1 and e_2 in $L(A\bar{z})$. To finish this case, we will display disjoint paths between the neighbors of e_1 and the neighbors of e_2 in $L(G) \setminus L(A\bar{z})$. To do this we will display a set of disjoint paths in $L(Az)$ and extend them via a certain matching to get the required paths.

In the table below we define some subsets $Eza, Ez\bar{a}\bar{b}, Ezb, Ez2, Ez3, Ez4$, $Fza, Fz\bar{a}\bar{b}, Fzb, Fz2, Fz3$ and $Fz4$ of $V(L(G))$ by specifying their members:

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
Eza	$\{a, x_0, ..., x_{m-2}\}$	$\{a, x_0, ..., x_{m-2}, z\} \setminus \{x_i\}$
$Ez\bar{a}\bar{b}$	$\{a, x_0, ..., x_{m-2}\}$ and $\{b, x_0, ..., x_{m-2}\}$	$\{x_0, ..., x_{m-2}, z\}$
Ezb	$\{b, x_0, ..., x_{m-2}\}$	$\{b, x_0, ..., x_{m-2}, z\} \setminus \{x_i\}$
$Ez2$	$\{a, x_0, ..., x_{m-2}, z\} \setminus \{x_i\}$	$\{b, x_0, ..., x_{m-2}, z\} \setminus \{x_i\}$
$Ez3$	$\{x_0, ..., x_{m-2}, z\}$	$\{a, x_0, ..., x_{m-2}, z\} \setminus \{x_0\}$ and $\{a, x_0, ..., x_{m-2}, z\} \setminus \{x_1\}$
$Ez4$	$\{b, x_0, ..., x_{m-2}, z\} \setminus \{x_i\}$	$\{x_0, ..., x_{m-2}, z\}$
Fza	$\{c, y_0, ..., y_{m-2}\}$	$\{c, y_0, ..., y_{m-2}, z\} \setminus \{y_i\}$
$Fz\bar{a}\bar{b}$	$\{c, y_0, ..., y_{m-2}\}$ and $\{d, y_0, ..., y_{m-2}\}$	$\{y_0, ..., y_{m-2}, z\}$
Fzb	$\{d, y_0, ..., y_{m-2}\}$	$\{d, y_0, ..., y_{m-2}, z\} \setminus \{y_i\}$

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
$Fz2$	$\{c, y_0, ..., y_{m-2}, z\} \setminus \{y_i\}$	$\{d, y_0, ..., y_{m-2}, z\} \setminus \{y_i\}$
$Fz3$	$\{y_0, ..., y_{m-2}, z\}$	$\{c, y_0, ..., y_{m-2}, z\} \setminus \{y_0\}$ and $\{c, y_0, ..., y_{m-2}, z\} \setminus \{y_1\}$
$Fz4$	$\{d, y_0, ..., y_{m-2}, z\} \setminus \{y_i\}$	$\{y_0, ..., y_{m-2}, z\}$

where $i \in \{0, 1, ..., m-2\}$.

Table 3 : The sets $Eza, Ez\bar{a}\bar{b}, Ezb, Ez2, Ez3, Ez4,\ Fza, Fz\bar{a}\bar{b}, Fzb, Fz2, Fz3$ and $Fz4$

The neighbors of e_1 in $L(G) \setminus L(A\bar{z})$ are the vertices in $Eza \cup Ez\bar{a}\bar{b} \cup Ezb$ and the neighbors of e_2 in $L(G) \setminus L(A\bar{z})$ are the vertices in $Fza \cup Fz\bar{a}\bar{b} \cup Fzb$. By construction, there is a matching in $L(G)$ that matches the vertices in $Eza, Ez\bar{a}\bar{b}, Ezb, Fza, Fz\bar{a}\bar{b}$ and Fzb to the vertices in $Ez2, Ez3, Ez4, Fz2, Fz3$ and $Fz4$, respectively. Let $Ez_T = Ez2 \cup Ez3 \cup Ez4$ and $Fz_T = Fz2 \cup Fz3 \cup Fz4$.

We now display disjoint paths in $L(Az)$ from all vertices in Ez_T to all vertices in Fz_T. Note that the vertices in Ez_T and Fz_T are in $L(Az)$. Since $|Ez_T| = |Fz_T| = (m-1)+2+(m-1) = 2m \leq 2(m-1)+4-2 \leq 2(3)(m-1)-2 \leq 2(m-1)(n-m)-2 = \kappa(L(Az))$, by Theorem 2.5, there exist $2m$ disjoint paths in $L(Az)$ from all vertices in Ez_T to all vertices in Fz_T. Let Pz be the set of these paths. Extend the paths in Pz via the above matching to get the paths from all neighbors of e_1 to all neighbors of e_2 in $L(G) \setminus L(A\bar{z})$.

Finally we show that the extended paths are disjoint. By construction the vertices extended via the matching (all neighbors of e_1 and e_2 in $L(G) \setminus L(A\bar{z})$) are distinct and they are in $L(G) \setminus L(A\bar{z})$ but not in $L(Az)$, but the disjoint paths in Pz are in $L(Az)$. Therefore the extended paths from all neighbors of e_1 to all neighbors of e_2 in $L(G) \setminus L(A\bar{z})$ are disjoint as required.

Case 3.2 : $\{a, b, c, d, x_0, ..., x_{m-2}, y_0, ..., y_{m-2}\} = S$.

Case 3.2.1 : $\{a, b\} \cap \{y_0, ..., y_{m-2}\} = \phi$ and $\{c, d\} \cap \{x_0, ..., x_{m-2}\} = \phi$.

Then $n = 2m+2$, $\{z_0, ..., z_{n-m-2}\} = \{c, d, y_0, ..., y_{m-2}\}$ and $\{w_0, ..., w_{n-m-2}\} = \{a, b, x_0, ..., x_{m-2}\}$. Without loss of generality, let $z_0 = c$, $z_1 = d$, $w_0 = a$, $w_1 = b$, and for $i \in \{2, ..., n-m-2\}$, let $z_i = y_{i-2}$ and $w_i = x_{i-2}$. Then e_2 is a vertex in $L(A\bar{a})$. Therefore it has $2m(n-m-1)-2$ neighbors in $L(A\bar{a})$. Let $F\bar{a}$ be the set of these neighbors.

Let $\oplus$ be the addition modulo $m-1$. In the table below we define some subsets $Ea1, Ea2, Ea3, Eb1, Eb2, Ea11, Ea12, Ea.b21, Ea31, Ea111, Ea121, Fa1, Fa2, Fa3, Fa11, Fa21, Fa31, Fa111, Fa211, Fa311$ and $Fa312$ of $V(L(G))$ by specifying their members:

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
$Ea1$	$\{a, x_0, ..., x_{m-2}\}$	$\{a, x_0, ..., x_{m-2}, z_j\} \setminus \{x_i\}$
$Ea2$	$\{a, x_0, ..., x_{m-2}\}$	$\{a, b, x_0, ..., x_{m-2}\} \setminus \{x_i\}$
$Ea3$	$\{a, x_0, ..., x_{m-2}\}$	$\{x_0, ..., x_{m-2}, z_j\}$
$Eb1$	$\{b, x_0, ..., x_{m-2}\}$	$\{b, x_0, ..., x_{m-2}, z_j\} \setminus \{x_i\}$ and $\{x_0, ..., x_{m-2}, z_j\}$
$Eb2$	$\{b, x_0, ..., x_{m-2}\}$	$\{a, b, x_0, ..., x_{m-2}\} \setminus \{x_i\}$

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
$Ea11$	$\{a, x_0, ..., x_{m-2}, z_0\} \setminus \{x_0\}$ and $\{a, x_0, ..., x_{m-2}, z_1\} \setminus \{x_0\}$	$\{a, b, x_0, ..., x_{m-2}, z_0\} \setminus \{x_0, x_1\}$ and $\{a, b, x_0, ..., x_{m-2}, z_1\} \setminus \{x_0, x_1\}$
$Ea12$	$\{a, x_0, ..., x_{m-2}, z_j\} \setminus \{x_i\}$ except when $i = 0$ and $j \in \{0, 1\}$	$\{x_0, ..., x_{m-2}, z_j, z_{j\oplus 1}\} \setminus \{x_i\}$ except when $i = 0$ and $j \in \{0, 1\}$
$Ea31$	$\{x_0, ..., x_{m-2}, z_j\}$	$\{b, x_0, ..., x_{m-3}, z_j\}$
$Ea.b21$	$\{a, b, x_0, ..., x_{m-2}\} \setminus \{x_i\}$	$\{a, b, x_0, ..., x_{m-2}, y_{m-2}\} \setminus \{x_i, x_{i\oplus 1}\}$ and $\{a, b, x_0, ..., x_{m-2}, y_{m-3}\} \setminus \{x_i, x_{i\oplus 1}\}$
$Ea111$	$\{a, b, x_0, ..., x_{m-2}, z_0\} \setminus \{x_0, x_1\}$ and $\{a, b, x_0, ..., x_{m-2}, z_1\} \setminus \{x_0, x_1\}$	$\{a, b, x_1, ..., x_{m-2}\}$
$Ea121$	$\{x_0, ..., x_{m-2}, z_j, z_{j\oplus 1}\} \setminus \{x_i\}$ except when $i = 0$ and $j \in \{0, 1\}$	$\{b, x_0, ..., x_{m-2}, z_j\} \setminus \{x_i\}$ except when $i = 0$ and $j \in \{0, 1\}$
$Fa1$	$\{c, y_0, ..., y_{m-2}\}$	$\{a, c, y_0, ..., y_{m-2}\} \setminus \{y_i\}$
$Fa2$	$\{d, y_0, ..., y_{m-2}\}$	$\{a, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$
$Fa3$	$\{c, y_0, ..., y_{m-2}\}$ and $\{d, y_0, ..., y_{m-2}\}$	$\{a, y_0, ..., y_{m-2}\}$
$Fa11$	$\{a, c, y_0, ..., y_{m-2}\} \setminus \{y_i\}$	$\{a, b, c, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$
$Fa21$	$\{a, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$	$\{a, b, d, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$
$Fa31$	$\{a, y_0, ..., y_{m-2}\}$	$\{a, b, y_0, ..., y_{m-2}\} \setminus \{y_0\}$ and $\{a, b, y_0, ..., y_{m-2}\} \setminus \{y_1\}$
$Fa111$	$\{a, b, c, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$	$\{a, b, x_i, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$
$Fa211$	$\{a, b, d, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$	$\{a, b, x_i, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$
$Fa311$	$\{a, b, y_0, ..., y_{m-2}\} \setminus \{y_0\}$	$\{a, b, x_0, y_0, ..., y_{m-2}\} \setminus \{y_0, y_1\}$
$Fa312$	$\{a, b, y_0, ..., y_{m-2}\} \setminus \{y_1\}$	$\{a, b, x_1, y_0, ..., y_{m-2}\} \setminus \{y_1, y_{1\oplus 1}\}$

where $i \in \{0, 1, ..., m-2\}$ and $j \in \{0, 1, ..., n-m-2\}$.

Table 4 : The sets $Ea1, Ea2, Ea3, Eb1, Eb2, Ea11, Ea12, Ea.b21, Ea31, Ea111, Ea121, Fa1, Fa2, Fa3, Fa11, Fa21, Fa31, Fa111, Fa211, Fa311$ and $Fa312$

The neighbors of e_1 in $L(G)$ are the vertices in $Ea1 \cup Ea2 \cup Ea3 \cup Eb1 \cup Eb2$ and the neighbors of e_2 in $L(G)$ are the vertices in $Fa1 \cup Fa2 \cup Fa3 \cup F\bar{a}$. To get disjoint paths between all neighbors of e_1 and all neighbors of e_2 in $L(G)$, we will display a set of disjoint paths and extend them, if necessary, via a certain matching to get the required paths.

By construction, there is a matching in $L(G)$ that matches the vertices in $Ea1, Ea2 \cup Eb2, Ea3, Fa1, Fa2$ and $Fa3$ to the vertices in $Ea11 \cup Ea12, Ea.b21, Ea31, Fa11, Fa21$ and $Fa31$, respectively, and there is another matching in $L(G)$ that matches the vertices in $Ea11, Ea12,, Fa11, Fa21$ and $Fa31$ to the vertices in $Ea111, Ea121, Fa111, Fa211$ and $Fa311 \cup Fa312$, respectively. Let $Fab = Fa111 \cup Fa211 \cup Fa311 \cup Fa312$, $Eab = Ea111 \cup Ea.b21$ and $E\bar{a} = Ea121 \cup Ea31 \cup Eb1$.

We now display disjoint paths from all vertices in $Eab \cup E\bar{a}$ to all vertices in $Fab \cup F\bar{a}$. Note that the vertices in Eab and Fab are in $L(Aab)$ and the vertices in $E\bar{a}$ and $F\bar{a}$ are in $L(A\bar{a})$.

Since $m \geq 3$ and $n = 2m+2$, we have $|Eab| = |Fab| = 2+(m-1)+(m-1) = 2m < 2m+2 = 2(m+2)-2 \leq 2(m-2)(m+2)-2 = 2(m-2)(n-m)-2 = \kappa(L(Aab))$, by Theorem 2.5 there exist $2m$ disjoint paths in $L(Aab)$ from all vertices in Eab to all vertices in Fab. Let Pab be the set of these disjoint paths.

Since $|E\bar{a}| = |F\bar{a}| = 2k(n-k-1)-2 = \kappa(L(A\bar{a}))$, by Theorem 2.5 there exist $2k(n-k-1)-2$ disjoint paths in $L(A\bar{a})$ from all vertices in $E\bar{a}$ to all vertices in $F\bar{a}$. Let $P\bar{a}$ be the set of these disjoint paths.

Therefore, the paths in $Pab \cup P\bar{a}$ are paths from all vertices in $Eab \cup E\bar{a}$ to all vertices in $Fab \cup F\bar{a}$. Among these paths, only the vertices in $Eab \cup Ea121 \cup Ea31$ are the end of the paths that do not begin at neighbors of e_1, and only the vertices in Fab are the end of the paths that do not end at neighbors of e_2. Extend the paths via the above matchings to get walks from neighbors of e_1 to neighbors of e_2. The paths in $Pab \cup P\bar{a}$ with the extension are the walks from all neighbors of e_1 to all neighbors of e_2 in $L(G)$.

Finally we show that the above walks are disjoint paths. By construction the vertices extended via the matching (the vertices in $Ea12$, $Ea3$, $Eb2$, $Fa1$, $Fa2$, $Fa3$, $Ea1$, $Ea11$, $Ea2$, $Fa11$, $Fa21$ and $Fa31$) are all distinct and they are in neither $L(Aab)$ nor $L(A\bar{a})$, but the disjoint paths in Pab and $P\bar{a}$ are in disjoint induced subgraphs $L(Aab)$ and $L(A\bar{a})$, respectively. Therefore the specified walks from all neighbors of e_1 to all neighbors of e_2 in $L(G)$ are disjoint paths as required.

Case 3.2.2 : $\{a,b\} \cap \{y_0,...,y_{m-2}\} \neq \phi$ or $\{c,d\} \cap \{x_0...,x_{m-2}\} \neq \phi$.

In this case, we have $n = 2m+1$ or $n = 2m$. Without loss of generality, let $c = x_{m-2}$ and $b \notin \{y_0,...,y_{m-2}\}$. Then $b \in \{w_0,...,w_{n-m-2}\}$ and e_1 is in $L(Ac)$. Therefore there exists $2(m-1)(n-m)-2$ vertices which are the neighbors of e_1 in $L(Ac)$. Let Ec be the set of these vertices.

Fix $q \in S \setminus \{b,c,d,y_0,...,y_{m-2}\}$. In the table below we define some subsets $E\bar{c}1, E\bar{c}2, E\bar{c}3, E\bar{c}11, E\bar{c}21, E\bar{c}31, E\bar{c}111, Fd1, Fd2, Fd3, Fc1, Fc2, Fd11, Fd12,$ $Fc.d21, Fc.d22, Fd121$ and $Fc.d211$ of $V(L(G))$ by specifying their members:

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
$E\bar{c}1$	$\{a,x_0,...,x_{m-2}\}$	$\{a,x_0,...,x_{m-3},z_j\}$
$E\bar{c}2$	$\{b,x_0,...,x_{m-2}\}$	$\{b,x_0,...,x_{m-3},z_j\}$
$E\bar{c}3$	$\{a,x_0,...,x_{m-2}\}$ and $\{b,x_0,...,x_{m-2}\}$	$\{a,b,x_0,...,x_{m-3}\}$
$E\bar{c}11$	$\{a,x_0,...,x_{m-3},z_j\}$	$\{a,b,x_0,...,x_{m-3},z_j\} \setminus \{x_0\}$
$E\bar{c}21$	$\{b,x_0,...,x_{m-3},z_j\}$	$\{a,b,x_0,...,x_{m-3},z_j\} \setminus \{x_0\}$
$E\bar{c}31$	$\{a,b,x_0,...,x_{m-3}\}$	$\{b,x_0,...,x_{m-3},z_0\}$ and $\{b,x_0,...,x_{m-3},z_1\}$
$E\bar{c}111$	$\{a,b,x_0,...,x_{m-3},z_j\} \setminus \{x_0\}$	$\{a,b,x_0,...,x_{m-3}\}$
$Fc1$	$\{c,y_0,...,y_{m-2}\}$	$\{c,y_0,...,y_{m-2},w_j\} \setminus \{y_i\}$ and $\{c,d,y_0,...,y_{m-2}\} \setminus \{y_i\}$
$Fc2$	$\{c,y_0,...,y_{m-2}\}$	$\{y_0,...,y_{m-2},w_j\}$
$Fd1$	$\{d,y_0,...,y_{m-2}\}$	$\{d,y_0,...,y_{m-2},w_j\} \setminus \{y_i\}$
$Fd2$	$\{d,y_0,...,y_{m-2}\}$	$\{y_0,...,y_{m-2},w_j\}$
$Fd3$	$\{d,y_0,...,y_{m-2}\}$	$\{c,d,y_0,...,y_{m-2}\} \setminus \{y_i\}$

Set	vertices in $L(G)$ induced by the edges in G that join	
	the vertices	to the vertices
$Fc.d21$	$\{y_0, ..., y_{m-2}, w_j\}; w_j \neq b$	$\{b, y_0, ..., y_{m-2}, w_j\} \setminus \{y_0\}$ and $\{b, y_0, ..., y_{m-2}, w_j\} \setminus \{y_1\}; w_j \neq b$
$Fc.d22$	$\{b, y_0, ..., y_{m-2}\}$	$\{d, b, y_0, ..., y_{m-2}\} \setminus \{y_0\}$ and $\{d, b, y_0, ..., y_{m-2}\} \setminus \{y_1\}$
$Fd11$	$\{d, b, y_0, ..., y_{m-2}\} \setminus \{y_0\}$ and $\{d, b, y_0, ..., y_{m-2}\} \setminus \{y_1\}$	$\{b, y_0, ..., y_{m-2}, q\} \setminus \{y_0\}$ and $\{b, y_0, ..., y_{m-2}, q\} \setminus \{y_1\}$
$Fd12$	$\{d, y_0, ..., y_{m-2}, w_j\} \setminus \{y_i\}$ except when $w_j = b$ and $i \in \{0, 1\}$	$\{c, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$
$Fd31$	$\{c, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$	$\{b, c, d, y_0, ..., y_{m-2}\} \setminus \{y_i, y_{i\oplus 1}\}$
$Fc.d211$	$\{b, y_0, ..., y_{m-2}, w_j\} \setminus \{y_0\}$ and $\{b, y_0, ..., y_{m-2}, w_j\} \setminus \{y_1\}; w_j \neq b$	$\{b, y_0, ..., y_{m-2}\}$
$Fd121$	$\{c, d, y_0, ..., y_{m-2}\} \setminus \{y_i\}$	$\{c, y_0, ..., y_{m-2}, w_j\} \setminus \{y_i\}$ except when $w_j = b$ and $i \in \{0, 1\}$

where $i \in \{0, 1, ..., m-2\}$ and $j \in \{0, 1, ..., n-m-2\}$.

Table 5 : The sets $E\bar{c}1, E\bar{c}2, E\bar{c}3, E\bar{c}11, E\bar{c}21, E\bar{c}31, E\bar{c}111, Fd1, Fd2, Fd3, Fc1, Fc2, Fd11, Fd12, Fc.d21, Fc.d22, Fd121$ and $Fc.d211$

The neighbors of e_1 in $L(G)$ are the vertices in $Ec \cup E\bar{c}1 \cup E\bar{c}2 \cup E\bar{c}3$ and the neighbors of e_2 in $L(G)$ are the vertices in $Fd1 \cup Fd2 \cup Fd3 \cup Fc1 \cup Fc2$. To get disjoint paths between all neighbors of e_1 and all neighbors of e_2 in $L(G)$, we will display a set of disjoint paths and extend them, if necessary, via a certain matching to get the required paths.

By construction, there is a matching in $L(G)$ that matches the vertices in $E\bar{c}1, E\bar{c}2, E\bar{c}3, Fd1, Fd2 \cup Fc2$ and $Fd3$ to the vertices in $E\bar{c}11, E\bar{c}21, E\bar{c}31, Fd11 \cup Fd12, Fc.d21 \cup Fc.d22$ and $Fd31$, respectively, and there is another matching in $L(G)$ that matches the vertices in $E\bar{c}11, Fd12$ and $Fc.d21$ to the vertices in $E\bar{c}111, Fd121$ and $Fc.d211$, respectively. Let $Eb\bar{c} = E\bar{c}111 \cup E\bar{c}21 \cup E\bar{c}31$, $Fb\bar{c} = Fd11 \cup Fc.d22 \cup Fc.d211$ and $Fc = Fc1 \cup Fd31 \cup Fd121$.

We now display disjoint paths from all vertices in $Eb\bar{c} \cup Ec$ to all vertices in $Fb\bar{c} \cup Fc$. Note that, the vertices in Ec and Fc are in $L(Ac)$ and the vertices in $Eb\bar{c}$ and $Fb\bar{c}$ are in $L(Ab\bar{c})$.

Since $m \geq 3$ and $n \geq 2m$, we have $|Eb\bar{c}| = |Fb\bar{c}| = 2(n-m) \leq 2(n-m) + 2(n-m) - 2m \leq 2(m-1)(n-m) - 2(m-1) - 2 = 2(m-1)(n-m-1) - 2 = \kappa(L(Ab\bar{c}))$, by Theorem 2.5, there exist $2(n-m)$ disjoint paths in $L(Ab\bar{c})$ from all vertices in $Eb\bar{c}$ to all vertices in $Fb\bar{c}$. Let $Pb\bar{c}$ be the set of these disjoint paths.

Since $|Ec| = |Fc| = 2(m-1)(n-m) - 2 = \kappa(L(Ac))$, by Theorem 2.5, there exist $2(m-1)(n-m) - 2$ disjoint paths in $L(Ac)$ from all vertices in Ec to all vertices in Fc. Let Pc be the set of these disjoint paths.

Therefore, the paths in $Pb\bar{c} \cup Pc$ are paths from all vertices in $Eb\bar{c} \cup Ec$ to all vertices in $Fb\bar{c} \cup Fc$. Among these paths, only the vertices in $Eb\bar{c}$ are the end of the paths that do not begin at neighbors of e_1, and only the vertices in $Fb\bar{c} \cup Fd31 \cup Fd121$ are the end of the paths that do not end at neighbors of e_2. Extend these paths via the above matchings to get walks from neighbors of e_1 to neighbors of e_2. The paths in $Pb\bar{c} \cup Pc$ with the extension are the walks from all neighbors of e_1 to all neighbors of e_2 in $L(G)$.

Finally we show that the above walks are disjoint paths. By construction the vertices extended via the matching (the vertices in $E\bar{c}1$, $E\bar{c}2$, $E\bar{c}3$, $E\bar{c}11$, $Fc2$, $Fd1$, $Fd2$, $Fd3$, $Fc.d21$ and $Fd12$) are all distinct and they are in neither $L(Ab\bar{c})$ nor $L(Ac)$ but the disjoint paths in $Pb\bar{c}$ and Pc are in disjoint induced subgraphs $L(Ab\bar{c})$ and $L(Ac)$ respectively. Therefore the specified walks from all neighbors of e_1 to all neighbors of e_2 in $L(G)$ are disjoint paths as required. □

By Proposition 3.4, Lemma 4.1 and Lemma 4.2, we obtain the connectivity of the line graphs of Johnson graphs.

Theorem 4.3. For a Johnson graph G with degree k, we have $\kappa(L(G)) = 2k - 2$.

The last theorem is the characterization of the graphs whose line graphs are Johnson graphs.

Theorem 4.4. Let G be a graph where $L(G)$ is a Johnson graph different from K_3. Then $L(G)$ is $J(n, 1)$ or $J(n, 2)$ for some positive integer n. Moreover, G is $K_{1,n}$ if $L(G) \cong J(n, 1)$ and G is K_n if $L(G) \cong J(n, 2)$.

Proof. Let $L(G)$ be $J(n, m)$ for some positive integers n and m where $J(n, m) \neq J(3, 1)$. Let $S = \{1, 2, 3, ..., n\}$ be the alphabet set of this Johnson graph.
Case 1 : $m \geq 3$. Then $n \geq 6$. For any distinct elements $x_1, ..., x_{m-3}$ in $S \setminus \{1, 2, ..., 6\}$, observe that $\{1, 2, 3, x_1, ..., x_{m-3}\}$, $\{1, 2, 4, x_1, ..., x_{m-3}\}$, $\{1, 3, 5, x_1, ..., x_{m-3}\}$ and $\{2, 3, 6, x_1, ..., x_{m-3}\}$ induce a subgraph $K_{1,3}$ in $L(G)$. This contradicts Theorem 2.9. Hence, this case does not occur.
Case 2 : $m = 1$. Then $L(G)$ is a complete graph. Hence, G is $K_{1,n}$ by Theorem 2.8.
Case 3 : $m = 2$. Then $L(G)$ is $J(n, 2)$. Let $J = L(G)$, and for each $x \in S$, let A_x be the set of all 2-subsets of S which contain x. Then the induced subgraph $J[A_x]$ is a clique of J for all $x \in S$.

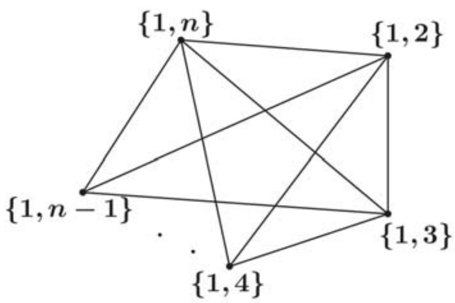


Figure 24 : Induced subgraph $J[A_1]$ in $L(G)$

Observe that a vertex $\{a, b\}$ of J is in $J[A_a]$ and $J[A_b]$, and an edge of J joining vertices $\{u, v\}$ and $\{u, w\}$ is in $J[A_u]$.

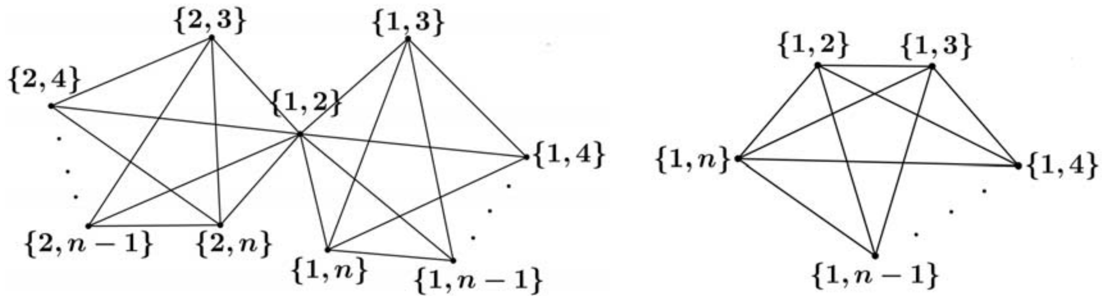
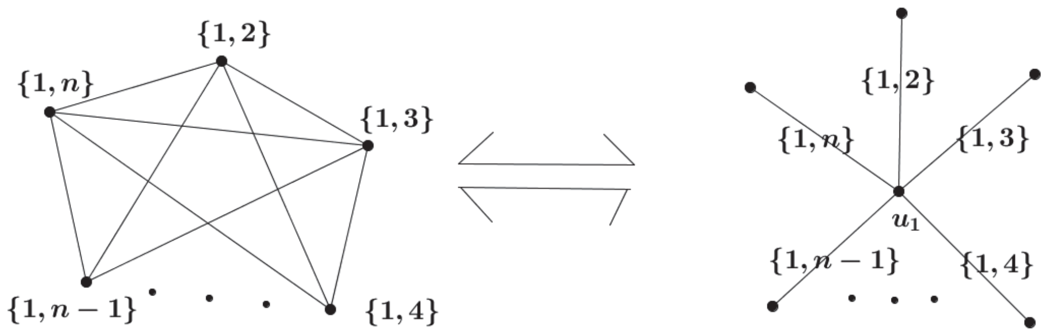


Figure 25 : The vertex $\{1, 2\}$ is in $J[A_1]$ and $J[A_2]$ and the edge from $\{1, 2\}$ to $\{1, 3\}$ is in $J[A_1]$ in $L(G)$

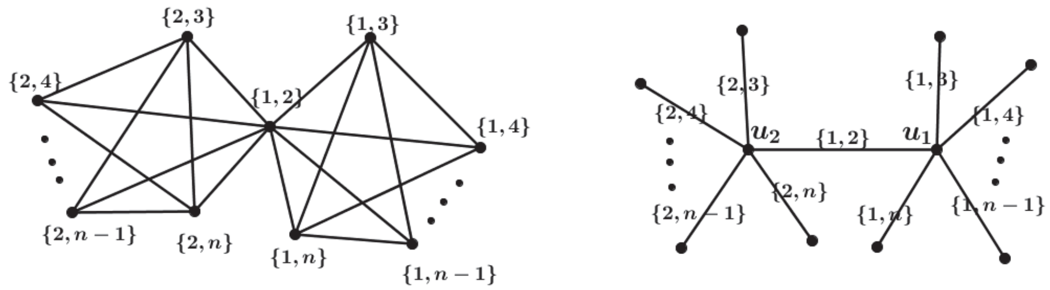
Hence $\{J[A_x]|x \in S\}$ is the Krauz partition of J. By Theorem 2.10, each $J[A_x]$ is induced by the star at a vertex in G, and $|V(G)| = |S| = n$. For each $x \in S$, let u_x be the vertex in G whose star induces $J[A_x]$.



$L(G)$ G

Figure 26 : The star of u_1 in G induces $J[A_1]$ in $L(G)$

Then for a vertex $\{x, y\}$ in J, the end vertices of the edge $\{x, y\}$ in G are u_x and u_y.



$L(G)$ G

Figure 27 : The vertex $\{1, 2\}$ in $L(G)$ is in $J[A_1]$ and $J[A_2]$. Therefore the edge $\{1, 2\}$ in G is in the stars of u_1 and u_2. Thus, u_1 and u_2 are the end vertices of the edge $\{1, 2\}$ in G.

For each $x \in S$, since $A_x = \{\{x, y\}|y \in S \text{ and } y \neq x\}$ and $J[A_x]$ is induced by the star at u_x in G, it follows that u_x is adjacent to u_y for all $y \in S$ such that $y \neq x$. That is each vertex of G is adjacent to all other vertices. Hence, G is K_n as required. □

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