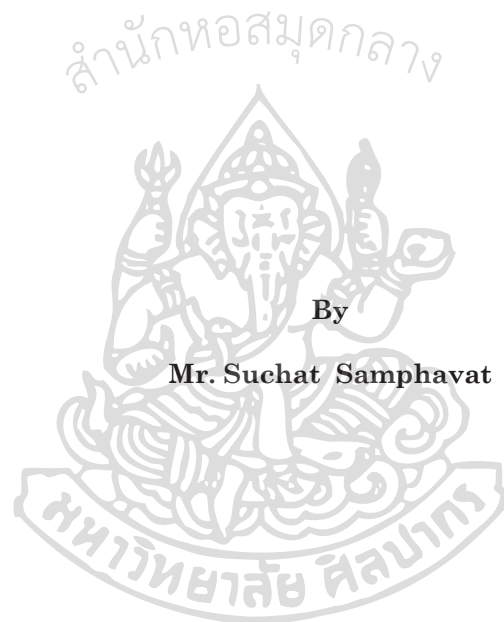




**DUALITY OF SEQUENCE SPACES OF INFINITE MATRICES**



**A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree**

**Master of Science Program in Mathematics**

**Department of Mathematics**

**Graduate School, Silpakorn University**

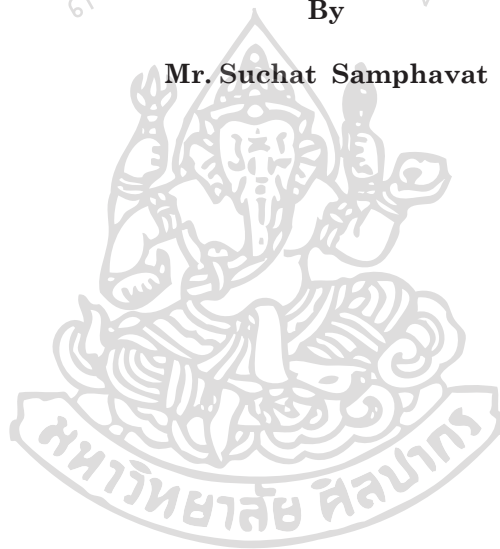
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**DUALITY OF SEQUENCE SPACES OF INFINITE MATRICES**

สำนักหอสมุดกลาง  
By

**Mr. Suchat Samphavat**



**A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree**

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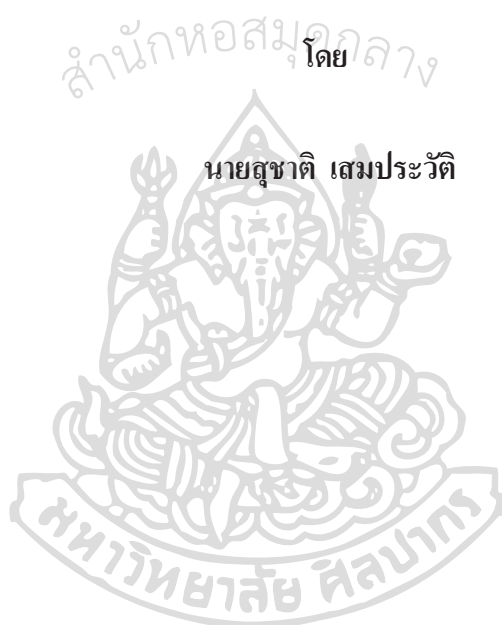
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ภาวะคู่กันของปริภูมิลำดับของแมทริกซ์อนันต์



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์

ภาควิชาคณิตศาสตร์

บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ปีการศึกษา 2554

ลิขสิทธิ์ของบัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Duality of sequence spaces of infinite matrices” submitted by MR.Suchat Samphavat as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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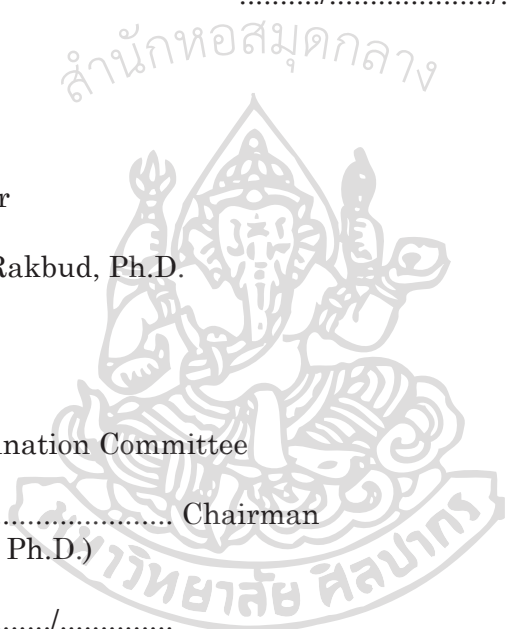
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52305204 : MAJOR : MATHEMATICS

KEY WORD : BANACH SPACE / DUALITY /  $l^1$  DIRECT- SUM

SUCHAT SAMPHAVAT : DUALITY OF SEQUENCE SPACES OF INFINITE MATRICES. THESIS ADVISOR : JITTI RAKBUD, Ph.D.. 31 pp.

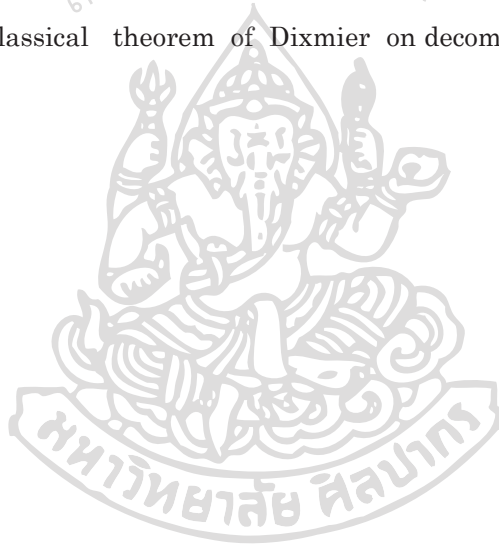
In this thesis, we define, for each  $1 \leq r < \infty$ , the set  $\mathfrak{F}^r$  of infinite complex matrices as follows:

$$\mathfrak{F}^r := \left\{ \left[ a_{ji}^{(k)} \right]_{k=1}^{\infty} : \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right] \in B(l^2) \right\}.$$

We first show as a preliminary that equipped with the norm

$$\left\| \left[ a_{ji}^{(k)} \right]_{k=1}^{\infty} \right\|_r := \left\| \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right] \right\|^{1/r},$$

the set  $\mathfrak{F}^r$  is a Banach space. The main goal of this research is to decompose the dual  $(\mathfrak{F}^r)^*$  of  $\mathfrak{F}^r$  as an  $l^1$  direct-sum of its two closed subspaces by a way analogous to the classical theorem of Dixmier on decomposing the dual  $B(l^2)^*$  of  $B(l^2)$ .



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Department of Mathematics

Graduate School, Silpakorn University

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คำสำคัญ : ปริภูมิบานาค/ ภาวะคู่กัน/ $l^1$  ผลบวกตรง

สุชาติ เสมประวัตติ : ภาวะคู่กันของปริภูมิลำดับของเมทริกซ์อนันต์. อาจารย์ที่ปรึกษา  
วิทยานิพนธ์ : อ.ดร.จิตติ รักรบุตร. 31 หน้า.

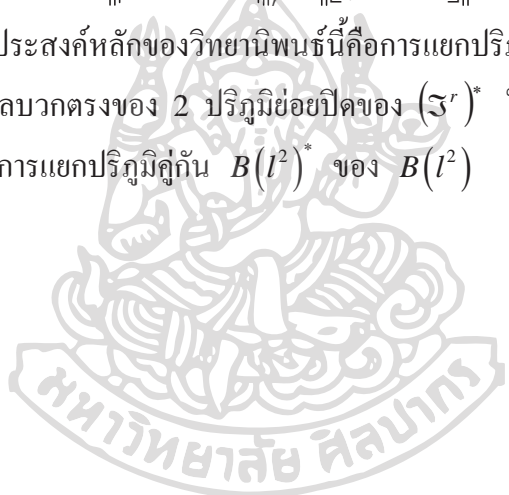
ในวิทยานิพนธ์นี้ เรานิยามเซต  $\mathfrak{S}^r$  ของเมทริกซ์อนันต์ของจำนวนเชิงซ้อน สำหรับ  
 $1 \leq r < \infty$  ดังนี้

$$\mathfrak{S}^r := \left\{ \left[ a_{ji}^{(k)} \right]_{k=1}^{\infty} : \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right] \in B(l^2) \right\}$$

ในขั้นแรก เราได้แสดงว่าเซต  $\mathfrak{S}^r$  พร้อมด้วยนอร์มที่นิยามโดย

$$\left\| \left[ a_{ji}^{(k)} \right]_{k=1}^{\infty} \right\|_r := \left\| \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right] \right\|^{1/r}$$

เป็นปริภูมิบานาค จุดประสงค์หลักของวิทยานิพนธ์นี้คือการแยกปริภูมิคู่กัน  $(\mathfrak{S}^r)^*$  ของ  $\mathfrak{S}^r$  เป็น  
ส่วน ในรูปของ  $l^1$  ผลบวกตรงของ 2 ปริภูมิย่อยปิดของ  $(\mathfrak{S}^r)^*$  ในทำนองเดียวกันกับทฤษฎีบท  
ของดิซซ์เมียร์ที่ว่าด้วยการแยกปริภูมิคู่กัน  $B(l^2)^*$  ของ  $B(l^2)$



ภาควิชาคณิตศาสตร์

บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร

ลายมือชื่อนักศึกษา.....

ปีการศึกษา 2554

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ .....

## ACKNOWLEDGEMENTS

This thesis has been completed by the involvement of people about whom I would like to mention here.

I would like to thank Dr. Jitti Rakbud, my advisor, for his valuable suggestions and excellent advices throughout the study with great attention.

I would like to thank Dr. Somjate Chaiya and Asst. Prof. Dr. Poom Kumam for their valuable comments and suggestions.

Finally, I would like to thank the centre of excellence in mathematics, the commission on higher education, Thailand, for the financial support .



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# Chapter 1

## Introduction and Preliminaries

A beautiful decomposition of the dual  $\mathcal{B}(l^2)^*$  of the Banach algebra  $\mathcal{B}(l^2)$  of bounded linear operators on  $l^2$ , under the usual multiplication, was established by Dixmier (see [5] and [6]). He proved that every bounded linear functional  $f$  in  $\mathcal{B}(l^2)^*$  can uniquely be decomposed as the sum  $f = g + h$  of two bounded linear functionals  $h$  and  $g$  on  $\mathcal{B}(l^2)$  such that  $g$  is a *trace* functional defined associated with a trace class operator  $T$  by  $g(S) = \text{trace}(ST)$  for all  $S \in \mathcal{B}(l^2)$ , and  $h$  vanishes on the ideal  $\mathcal{K}(l^2)$  of compact operators on  $l^2$ . The most interesting part of the theorem of Dixmier mentioned above is that the norm of the decomposition  $f = g + h$  of each  $f$  in  $\mathcal{B}(l^2)^*$  is additive, i.e.,  $\|f\| = \|g\| + \|h\|$ . A part of Schatten's theorem (see [12]) states that  $\mathcal{K}(l^2)^* \cong \mathcal{C}^1$ , where  $\mathcal{C}^1$  denotes the class of all trace class operators on  $l^2$ . From the theorem of Schatten, it can easily be deduced that the space of all trace functionals on  $\mathcal{B}(l^2)$  is isometrically isomorphic to the dual  $\mathcal{K}(l^2)^*$  of  $\mathcal{K}(l^2)$ . Thus the theorem of Dixmier mentioned above can be symbolized as  $\mathcal{B}(l^2)^* = \mathcal{K}(l^2)^* \oplus_1 \mathcal{K}(l^2)_s$ , where  $\mathcal{K}(l^2)_s$  denotes the space of all linear functionals on  $\mathcal{B}(l^2)$  vanishing on  $\mathcal{K}(l^2)$ , which are called *singular functionals* on  $\mathcal{K}(l^2)$ , and the notation " $\oplus_1$ " is referred to as the  $l^1$  direct-sum.

Let  $1 \leq p, q, r < \infty$ . An infinite scalar matrix  $A = [a_{jk}]$  is said to *define* a linear operator from  $l^p$  into  $l^q$  if for every  $x = \{x_k\}_{k=1}^{\infty}$  in  $l^p$  the series  $\sum_{k=1}^{\infty} a_{jk}x_k$  converges for all  $j$ , and the sequence  $Ax := \left\{ \sum_{k=1}^{\infty} a_{jk}x_k \right\}_{j=1}^{\infty}$  is a member of  $l^q$ . If a matrix  $A$  defines a linear operator from  $l^p$  into  $l^q$ , we then call the operator  $x \mapsto Ax$  the *linear operator defined by*  $A$ . In this case, it can be shown by the uniform boundedness principle that the linear operator defined by  $A$  is bounded. Let  $\mathcal{M}(l^p, l^q)$  be the set of all infinite matrices which define linear operators from  $l^p$  into  $l^q$ . For each matrix  $A$ , we call  $A$  a *bounded matrix* and define  $\|A\|$  to be the norm of the linear operator defined by  $A$  if  $A \in \mathcal{M}(l^p, l^q)$  and call  $A$  an *unbounded matrix* and define  $\|A\|$  to be  $\infty$  otherwise. It is well-known that  $\mathcal{M}(l^p, l^q)$  is a Banach space under the norm  $\|\cdot\|$ . Indeed, it coincides with the set of matrix representations of all bounded linear operators from  $l^p$  into  $l^q$  with respect to the standard Schauder bases of  $l^p$  and  $l^q$ , which is isometrically isomorphic to the Banach space  $\mathcal{B}(l^p, l^q)$  of all bounded linear operators from  $l^p$  into  $l^q$ . A matrix  $A$  is called a *compact matrix* if the linear operator defined by  $A$  is a compact operator.

For each matrix  $A = [a_{ji}]$  and positive integer  $n$ , let  $A_{n_s} = [b_{ji}]$  be the matrix with  $b_{ji} = a_{ji}$  for all  $1 \leq j, i \leq n$  and  $b_{ji} = 0$  otherwise, and let  $A_{n_r} = [c_{ji}]$  be the matrix with  $c_{ji} = a_{ji}$  for all  $j, i \geq n$  and  $c_{ji} = 0$  otherwise. The following are well-known facts about infinite matrices which are useful for the research.

**Theorem 1.1.**

- (1) If  $[a_{ji}]$  and  $[b_{ji}]$  are scalar matrices such that  $|a_{ji}| \leq b_{ji}$  for all  $j, i$ , then  $\|[a_{ji}]\| \leq \|[b_{ji}]\|$ .
- (2) A matrix  $A$  belongs to  $\mathcal{B}(l^p, l^q)$  if and only if  $\sup_n \|A_{n_s}\| < \infty$ .
- (3) For every matrix  $A$ ,  $\|A_{n_s}\| \nearrow \|A\|$ .
- (4) For each  $A \in \mathcal{B}(l^2)$  and positive integer  $n$ ,  $\|A_{n_s} + A_{n_r}\| = \max\{\|A_{n_s}\|, \|A_{n_r}\|\}$ .
- (5) A matrix  $A$  is compact as an operator on  $l^2$  if and only if  $\|A_{n_s} - A\| \rightarrow 0$ .

The Schur product or Hadamard product or entry-wise product of two scalar matrices  $A = [a_{jk}]$  and  $B = [b_{jk}]$  having the same size is defined by the matrix  $A \bullet B := [a_{jk}b_{jk}]$ . In [13], Schur proved that Banach space  $\mathcal{B}(l^2)$  is a commutative Banach algebra (without identity) under the operator norm and the Schur product multiplication. After that, Bennett extended in [1] the result of Schur referred to above. He showed for each  $1 \leq p, q < \infty$  that the Banach space  $\mathcal{B}(l^p, l^q)$  under the Schur product operation is also a Banach algebra. These beautiful results of Bennett motivated Chaisuriya and Ong [2] to study some classes of infinite matrices over Banach algebras with identity. In [2], for a fixed Banach algebra  $\mathcal{B}$  with identity and  $1 \leq p, q, r < \infty$ , the authors defined the class  $\mathcal{S}_{p,q}^r(\mathcal{B})$  of matrices  $A = [a_{jk}]$  over  $\mathcal{B}$  such that the absolute Schur  $r^{\text{th}}$ -power  $A^{[r]} := [\|a_{jk}\|^r]$  defines a linear operator from  $l^p$  into  $l^q$ . And then they proved that it is a Banach algebra under the absolute Schur  $r$ -norm defined by

$$\|A\|_{p,q,r} = \|A^{[r]}\|^{1/r}$$

and the Schur product, which is straightforwardly generalized to the setting of matrices over the Banach algebra  $\mathcal{B}$  by using the multiplication in  $\mathcal{B}$ . The authors also provided a beautiful relationship, which follows from the results of Schur and Bennett mentioned above, between the algebra  $\mathcal{B}(l^p, l^q)$  of all bounded operators from  $l^p$  into  $l^q$  and the algebra  $\mathcal{S}_{p,q}^r(\mathbb{C})$ . They found that  $\mathcal{B}(l^p, l^q)$  is contained in  $\mathcal{S}_{p,q}^r(\mathbb{C})$  as a non-closed ideal for all  $r \geq 2$ .

In [8], Livshits, Ong and Wang studied the duality of the absolute Schur algebras  $\mathcal{S}_{2,2}^r(\mathbb{C})$  by a way analogous to Dixmier's theorem and Schatten's theorem mentioned in the first paragraph. The authors defined the class  $\mathcal{K}^r$  of infinite matrices  $A$  such that  $A^{[r]}$  is compact as an operator on  $l^2$  for playing the role as the class  $\mathcal{K}(l^2)$  of all compact operators on  $l^2$ . They also constructed a class  $\mathcal{M}^r$  of infinite matrices for playing the role as the class  $\mathcal{C}^1$  of all trace class operators, which is known as the dual of  $\mathcal{K}(l^2)$ . They obtained that  $(\mathcal{K}^r)^* \cong \mathcal{M}^r$  and that each bounded

linear functional  $\varphi$  on  $\mathcal{S}_{2,2}^r(\mathbb{C})$  can uniquely be decomposed as the sum  $\varphi = \rho + \psi$ , where  $\rho$  is determined by a unique matrix in  $\mathcal{M}^r$  under a certain way and  $\psi$  is a singular functional on  $\mathcal{K}^r$ . Furthermore, the decomposition  $\varphi = \rho + \psi$  satisfies  $\|\varphi\| = \|\rho\| + \|\psi\|$ . Schatten's theorem also states that the trace class operators form a predual of  $\mathcal{B}(l^2)$ . An analogue of this result on the setting of Livshits, Ong and Wang:  $(\mathcal{M}^r)^* \cong \mathcal{S}_{2,2}^r(\mathbb{C})$ , was also obtained.

From the beautiful result of Chaisuriya and Ong that the absolute Schur algebra  $\mathcal{S}_{2,2}^2(\mathbb{C})$  contains  $\mathcal{B}(l^2)$  as a non-closed ideal, Rakbud and Ong defined three sequence spaces of matrices from  $\mathcal{S}_{2,2}^2(\mathbb{C})$  in [11] as follows:

$$\mathcal{O}_b = \left\{ \{A_k\}_{k=1}^\infty \subseteq \mathcal{S}_{2,2}^2(\mathbb{C}) : \text{the sequence } \left\{ \sum_{k=1}^n A_k^{[2]} \right\}_{n=1}^\infty \text{ is bounded in } \mathcal{B}(l^2) \right\},$$

$$\mathcal{O}_c = \left\{ \{A_k\}_{k=1}^\infty \subseteq \mathcal{S}_{2,2}^2(\mathbb{C}) : \text{the sequence } \left\{ \sum_{k=1}^n A_k^{[2]} \right\}_{n=1}^\infty \text{ converges in } \mathcal{B}(l^2) \right\},$$

and

$$\mathcal{O}_\kappa = \left\{ \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^\infty \subseteq \mathcal{S}_{2,2}^2(\mathbb{C}) : \text{the matrix } \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^2 \right] \in \mathcal{K}(l^2) \right\}.$$

The authors obtained the inclusion relation among these three spaces as follows:  $\mathcal{O}_\kappa \subsetneq \mathcal{O}_c \subsetneq \mathcal{O}_b$ . They defined naturally a norm on these three spaces by

$$\| \{A_k\}_{k=1}^\infty \| = \left( \sup_n \left\| \sum_{k=1}^n A_k^{[2]} \right\| \right)^{1/2}$$

and showed that all three sequence spaces equipped with this norm are Banach spaces. It was observed that because of the non-closedness of  $\mathcal{B}(l^2)$  in  $\mathcal{S}_{2,2}^2(\mathbb{C})$ , the restrictions of these sequence spaces to  $\mathcal{B}(l^2)$  are all not complete. The study on this paper was mainly focused on the sequence spaces  $\mathcal{O}_c$  and  $\mathcal{O}_\kappa$ . The authors studied sequential convergence in these two sequence spaces and duality and preduality of  $\mathcal{O}_\kappa$ .

From the idea of Rakbud and Ong referred to above, we obtain a way analogous to the classical sequence spaces  $l^p$  to define sequence spaces of infinite matrices as follows. Let  $\mathcal{M}_\infty$  be the vector space of all infinite complex matrices. For each  $1 \leq r < \infty$ , let

$$\mathcal{L}^r = \left\{ \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^\infty \subseteq \mathcal{M}_\infty : \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right] \in \mathcal{B}(l^2) \right\}.$$

In this research, we study some elementary properties and provide some results on duality of the sequence spaces  $\mathcal{L}^r$ . The main goal is to establish a decomposition theorem for the dual space  $(\mathcal{L}^r)^*$  of  $\mathcal{L}^r$  by a way analogous to the theorem of Dixmier mentioned in the first paragraph.

# Chapter 2

## Theoretical Background

In this chapter, we provide some theoretical background which is necessary for the research.

Throughout this thesis, we let  $\mathbb{C}$  and  $\mathbb{R}$  denote the sets of all complex numbers and real numbers respectively.

### 2.1 Banach Spaces

**Definition 2.1.1.** [9] Let  $X$  be a vector space over a scalar field  $\mathbb{K}$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ). A *norm* on  $X$  is a real-valued function  $\|\cdot\|$  on  $X$  satisfying the following properties:

- (i)  $\|x\| \geq 0$ ;
- (ii)  $\|x\| = 0$  if and only if  $x = 0$ ;
- (iii)  $\|\alpha x\| = |\alpha| \|x\|$ ;
- (iv)  $\|x + y\| \leq \|x\| + \|y\|$  (*Triangle inequality*),

where  $x$  and  $y$  are arbitrary vectors in  $X$  and  $\alpha$  is any scalar in  $\mathbb{K}$ . A *normed space* is a pair  $(X, \|\cdot\|)$  of a non-empty set  $X$  and a norm  $\|\cdot\|$  on  $X$ . It may be sometimes written just  $X$  as a normed space by omitting the norm on  $X$ .

**Definition 2.1.2.** [9] A sequence  $\{x_n\}_{n=1}^{\infty}$  in a normed space  $X$  is said to *converge* or to be *convergent* if there is a point  $x$  in  $X$  satisfying the following property: for any  $\epsilon > 0$ , there is a positive integer  $N$  such that

$$\|x - x_n\| < \epsilon \quad \text{for all } n \geq N.$$

In this situation, we write  $\lim_{n \rightarrow \infty} x_n = x$ , or simply  $x_n \rightarrow x$  and call  $x$  the *limit* of  $\{x_n\}_{n=1}^{\infty}$ .

**Definition 2.1.3.** [9] A sequence  $\{x_n\}_{n=1}^{\infty}$  in a normed space  $X$  is said to be *bounded* if there is a positive real number  $c$  such that  $\|x_n\| \leq c$  for all positive integer  $n$ .

**Definition 2.1.4.** [9] A sequence  $\{x_n\}_{n=1}^{\infty}$  in a normed space  $X$  is said to be a *Cauchy sequence* in  $X$  if for any  $\epsilon > 0$ , there is a positive integer  $N$  such that

$$\|x_m - x_n\| < \epsilon$$

for all  $m, n \geq N$ . A normed space  $X$  is said to be a *Banach space* if it is *complete* under the metric  $d$  defined by  $d(x, y) = \|x - y\|$ , that is, every Cauchy sequence converges to an element in  $X$ .

**Definition 2.1.5.** [9] Let  $X$  and  $Y$  be vector spaces over the same scalar field. A function  $T : X \rightarrow Y$  is said to be a *linear operator* or *linear function* or *linear transformation* if

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$$

for every  $x_1, x_2 \in X$  and any scalars  $\alpha$  and  $\beta$ .

**Definition 2.1.6.** [9] Let  $X$  and  $Y$  be normed spaces over the same scalar field. A linear operator  $T : X \rightarrow Y$  is said to be *bounded* if  $T(B)$  is bounded for all bounded subsets  $B$  of  $X$ .

**Definition 2.1.7.** Let  $T$  be a linear operator from a normed space  $X$  into a normed space  $Y$ . Then the range of  $T$  is denoted by  $\text{ran } T$ . We call the set  $\{x \in X : Tx = 0\}$  the *kernel* of  $T$  and denote by  $\ker T$ .

**Theorem 2.1.8.** [9] Let  $T : X \rightarrow Y$  be a linear operator from a normed space  $X$  into a normed space  $Y$ . Then the following are equivalent.

- (1)  $T$  is bounded.
- (2)  $T$  is continuous.
- (3) There is a constant  $M > 0$  such that  $\|Tx\| \leq M \|x\|$  for all  $x \in X$ .

Let  $\mathcal{B}(X, Y)$  be the set of all bounded linear operators from a normed space  $X$  into a normed space  $Y$ . We denote  $\mathcal{B}(X, X)$  by just  $\mathcal{B}(X)$ .

**Definition 2.1.9.** [9] Let  $X$  and  $Y$  be normed spaces. For each  $T$  in  $\mathcal{B}(X, Y)$ , the *norm* or *operator norm*  $\|T\|$  of  $T$  is the nonnegative real number  $\sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$ . The operator norm on  $\mathcal{B}(X, Y)$  is the map  $T \mapsto \|T\|$ .

From Theorem 2.1.8, the following corollary is immediately obtained.

**Corollary 2.1.10.** [9] If  $T$  is a bounded linear operator from a normed space  $X$  into a normed space  $Y$ , then  $\|Tx\| \leq \|T\| \|x\|$  for all  $x$  in  $X$ . Furthermore, the number  $\|T\|$  is the smallest nonnegative real number  $M$  such that  $\|Tx\| \leq M \|x\|$  for all  $x \in X$ .

**Definition 2.1.11.** [9] Let  $T$  be a linear operator from a normed space  $X$  onto a normed space  $Y$ . The operator  $T$  is an *isometric isomorphism* if  $\|T(x)\| = \|x\|$  whenever  $x \in X$ .

Notice that the condition  $\|T(x)\| = \|x\|$  for all  $x \in X$  implies  $T$  is an one-to-one function.

**Theorem 2.1.12.** [9] If  $X$  is a normed space and  $Y$  is a Banach space, then the set  $\mathcal{B}(X, Y)$  equipped with the operator norm is a Banach space.

**Theorem 2.1.13.** [9] (The Uniform Boundedness Principle) Let  $\mathcal{F}$  be a nonempty family of bounded linear operators from a Banach space  $X$  into a normed space  $Y$ . If  $\sup\{\|Tx\| : T \in \mathcal{F}\}$  is finite for each  $x$  in  $X$ , then  $\sup\{\|T\| : T \in \mathcal{F}\}$  is finite.

**Definition 2.1.14.** [9] A normed space  $X$  is said to be the *direct sum* of its two subspaces  $Y$  and  $Z$ , written by  $X = Y \oplus Z$ , if each  $x \in X$  has a unique representation of the form  $x = y + z$ , where  $y \in Y$  and  $z \in Z$ . If, in addition, the condition  $\|x\| = \|y\| + \|z\|$  is satisfied for all  $x \in X$ , we say specifically that  $X$  is the  $l^1$  *direct-sum* of  $Y$  and  $Z$  and write  $X = Y \oplus_1 Z$  in this situation.

**Theorem 2.1.15.** [9] Let  $X$  be a normed space and  $Y$  and  $Z$  be subspaces of  $X$ . Then  $X = Y \oplus Z$  if and only if for  $X \cap Y = \{0\}$  and for every  $x$  in  $X$ , there are  $y \in Y$  and  $z \in Z$  such that  $x = y + z$ .

**Definition 2.1.16.** [9] A *linear functional*  $f$  is a linear operator from a normed space  $X$  into the scalar field  $\mathbb{K}$  of  $X$ , where  $\mathbb{K}$  is regarded as a normed space under the usual norm on  $\mathbb{K}$ .

If  $X$  is a normed space, then the set of all bounded linear functionals on  $X$  is denoted by  $X^*$ . By Theorem 2.1.11, the normed space  $X^*$  is immediately a Banach space.

**Theorem 2.1.17.** [9] (Hahn-Banach extension theorem) Let  $X$  be a Banach space and  $Y$  a closed subspace of  $X$ . If  $f_0$  is a bounded linear functional on  $Y$ , then there is a unique bounded linear functional  $f$  on  $X$  such that  $f(x) = f_0(x)$  for all  $x \in Y$  and  $\|f\| = \|f_0\|$ .

**Definition 2.1.18.** [9] Let  $X$  be a normed space and  $Y$  a subspace of  $X$ . The *annihilator* of  $Y$ , denoted by  $Y^\perp$ , is the set  $\{f \in X^* : f(x) = 0 \text{ for all } x \in Y\}$ .

**Theorem 2.1.19.** [9] If  $X$  is a normed space and  $Y$  is a subspace of  $X$ , then  $Y^\perp$  is a closed subspace of  $X^*$ .

**Definition 2.1.20.** [9] Let  $X$  and  $Y$  be Banach spaces. A linear operator  $T : X \rightarrow Y$  is *compact* if  $\overline{T(B)}$  is compact for all bounded subset  $B$  of  $X$ . The set of all compact operators from  $X$  into  $Y$  will be denoted by  $\mathcal{K}(X, Y)$ . For the case where  $X = Y$ , we write  $\mathcal{K}(X)$  instead of  $\mathcal{K}(X, Y)$ .

**Proposition 2.1.21.** [9] Let  $X$  and  $Y$  be Banach spaces. Then the following hold.

- (1)  $\mathcal{K}(X, Y) \subseteq \mathcal{B}(X, Y)$ .
- (2)  $\mathcal{K}(X, Y)$  is a closed subspace of  $\mathcal{B}(X, Y)$ .
- (3) If  $X = Y$ , then  $\mathcal{K}(X)$  is an ideal of  $\mathcal{B}(X)$ .

**Definition 2.1.22.** [9] A linear operator  $T$  from a Banach space  $X$  into a Banach space  $Y$  is said to be of *finite rank* if  $T(X)$  is finite dimensional.

**Theorem 2.1.23.** [9] A finite rank operator from a Banach space  $X$  into a Banach space  $Y$  is bounded if and only if it is compact.

## 2.2 $l^p$ Spaces

**Definition 2.2.1.** [9] For  $1 \leq p \leq \infty$  and a sequence  $\{\lambda_k\}_{k=1}^{\infty}$  of complex numbers, the  $p$ -norm of  $\{\lambda_k\}_{k=1}^{\infty}$  is defined by

$$\|\{\lambda_k\}_{k=1}^{\infty}\|_p = \begin{cases} \left( \sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\{|\lambda_k| : k = 1, 2, 3, \dots\} & \text{if } p = \infty. \end{cases}$$

For each  $1 \leq p < \infty$ , let

$$l^p = \left\{ \{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sum_{k=1}^{\infty} |\lambda_k|^p < \infty \right\}$$

and

$$l^{\infty} = \{ \{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sup\{|\lambda_k| : k = 1, 2, 3, \dots\} < \infty \}.$$

**Theorem 2.2.2.** [9] (Hölder's inequality) For any  $1 \leq p \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and sequences  $\mathbf{x}$  and  $\mathbf{y}$  of complex numbers,  $\|\mathbf{xy}\|_1 \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$ .

In particular, Hölder's inequality is also called Cauchy-Schwartz's inequality when  $p = q = 2$ . From Hölder's inequality, the following Minkowski's inequality is obtained.

**Theorem 2.2.3.** [9] (Minkowski's inequality) For any  $1 \leq p \leq \infty$  and sequences  $\mathbf{x}$  and  $\mathbf{y}$  of complex numbers,  $\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$ .

**Theorem 2.2.4.** [9] For any  $1 \leq p \leq \infty$ , the set  $l^p$  endowed with the  $p$ -norm  $\|\cdot\|_p$  is a Banach space.

For  $1 \leq p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we define, for each  $\mathbf{x} = \{x_k\}_{k=1}^{\infty} \in l^q$ , the function  $f_{\mathbf{x}} : l^p \rightarrow \mathbb{C}$  by

$$f_{\mathbf{x}}(\{y_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} x_k y_k \text{ for all } \{y_k\}_{k=1}^{\infty} \in l^p.$$

By Hölder's inequality, we have that the function  $f_{\mathbf{x}}$  is well-defined.

**Theorem 2.2.5.** [9] Let  $1 \leq p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $l^q$  is isometrically isomorphic to  $(l^p)^*$  by the isomorphism defined by  $\mathbf{x} \mapsto f_{\mathbf{x}}$ .

The following result is closely related to the duality theorem stated above.

**Theorem 2.2.6.** [9] Let  $1 \leq p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then a sequence  $\{x_k\}_{k=1}^{\infty}$  of complex numbers belongs to  $l^q$  if and only if  $\{x_k y_k\}_{k=1}^{\infty}$  belongs to  $l^1$  for all  $\{y_k\}_{k=1}^{\infty}$  in  $l^p$ .

## 2.3 Hilbert Spaces

**Definition 2.3.1.** [4] Let  $\mathcal{H}$  be a vector space over a scalar  $\mathbb{K}$  ( $\mathbb{K}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ ), a *semi-inner product* on  $\mathcal{H}$  is a function  $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{K}$  having the following properties:

- (i)  $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ ;
- (ii)  $\langle x, x \rangle \geq 0$ ;
- (iii)  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ .

If  $\langle \cdot, \cdot \rangle$  has the following additional property:

- (iv) if  $\langle x, x \rangle = 0$ , then  $x = 0$ ,

we call  $\langle \cdot, \cdot \rangle$  an *inner product* on  $\mathcal{H}$ .

From (i), we have  $\langle 0, y \rangle = \langle 0x, y \rangle = 0 \langle x, y \rangle = 0$ , and similarly,  $\langle x, 0 \rangle = 0$ . In particular,  $\langle 0, 0 \rangle = 0$ . Hence if  $\langle \cdot, \cdot \rangle$  is an inner-product, then  $\langle x, x \rangle = 0$  if and only if  $x = 0$ . If  $\langle \cdot, \cdot \rangle$  is an inner-product on  $\mathcal{H}$ , then

$$\|x\| = \langle x, x \rangle^{1/2}$$



defines a norm on  $\mathcal{H}$ . We call a vector space  $\mathcal{H}$  equipped with an inner product on  $\mathcal{H}$  an *inner product space*. Every inner product space is a normed space under the norm defined by  $\|x\| = \langle x, x \rangle^{1/2}$ . If  $\mathcal{H}$  equipped with the norm  $\|\cdot\|$  is a Banach space, we call  $\mathcal{H}$  a *Hilbert space*.

Let, in the sequel,  $\mathcal{H}$  and  $\mathcal{L}$  be Hilbert spaces.

**Definition 2.3.2.** [4] If  $f, g \in \mathcal{H}$ , then  $f$  and  $g$  are *orthogonal* if  $\langle f, g \rangle = 0$ , in symbols,  $f \perp g$ . If  $A, B \subseteq \mathcal{H}$ , we say that  $A$  and  $B$  are *orthogonal* and write  $A \perp B$  provided  $f \perp g$  for every  $f$  in  $A$  and  $g$  in  $B$ . If  $A \subseteq \mathcal{H}$  and  $f \in \mathcal{H}$  satisfying  $\{f\} \perp A$ , then we write  $f \perp A$ . If  $A \subseteq \mathcal{H}$ , then the set  $A^\perp$  is defined by  $A^\perp = \{h \in \mathcal{H} : h \perp g \text{ for all } g \in A\}$ .

**Definition 2.3.3.** [4] An *orthonormal set* in  $\mathcal{H}$  is a subset  $\mathcal{E}$  of  $\mathcal{H}$  having the following properties:

- (i) for  $e \in \mathcal{E}$ ,  $\|e\| = 1$ ;
- (ii) if  $e_1, e_2 \in \mathcal{E}$  and  $e_1 \neq e_2$ , then  $e_1 \perp e_2$ .

An *orthonormal basis* for  $\mathcal{H}$  is a maximal orthonormal set.

**Proposition 2.3.4.** [4] If  $\mathcal{E}$  is an orthonormal set in  $\mathcal{H}$ , then there is an orthonormal basis for  $\mathcal{H}$  that contains  $\mathcal{E}$ .

**Theorem 2.3.5.** [4] If  $\mathcal{E}$  is an orthonormal set in  $\mathcal{H}$  and  $h \in \mathcal{H}$ , then  $\{e \in \mathcal{E} : \langle h, e \rangle \neq 0\}$  is countable.

**Theorem 2.3.6.** [4] Let  $\mathcal{E}$  be an orthonormal set in  $\mathcal{H}$ . Then the following statements are equivalent.

- (1)  $\mathcal{E}$  is an orthonormal basis.
- (2) If  $h \in \mathcal{H}$  and  $h \perp \mathcal{E}$ , then  $h = 0$ .
- (3)  $\bigvee \mathcal{E} = \mathcal{H}$ , where  $\bigvee \mathcal{E}$  is the smallest closed subspace of  $\mathcal{H}$  containing  $\mathcal{E}$ .
- (4)  $h = \sum \{\langle h, e \rangle e : e \in \mathcal{E}\}$  for all  $h \in \mathcal{H}$ , where  $\sum \{\langle h, e \rangle e : e \in \mathcal{E}\}$  denotes the limit of the net  $\left\{ \sum_{e \in F} \langle h, e \rangle e : F \text{ is a finite subset of } \mathcal{E} \right\}$ .

**Theorem 2.3.7.** Any two orthonormal bases of  $\mathcal{H}$  have the same cardinality.

**Definition 2.3.8.** [4] The *dimension* of  $\mathcal{H}$  is the cardinality of an orthonormal basis and is denoted by  $\dim \mathcal{H}$ .

**Definition 2.3.9.** [4] A subset  $D$  of  $\mathcal{H}$  is said to be *dense* in  $\mathcal{H}$  if  $\overline{D} = \mathcal{H}$ .  $\mathcal{H}$  is said to be *separable* if it has a countable subset which is dense in  $\mathcal{H}$ .

**Theorem 2.3.10.** [4] Let  $\mathcal{H}$  be an infinite dimensional Hilbert space. Then  $\mathcal{H}$  is separable if and only if  $\dim\mathcal{H} = \aleph_0$ , where  $\aleph_0$  is the cardinality of the set of all positive integers.

**Definition 2.3.11.** [4] A function  $u : \mathcal{H} \times \mathcal{L} \rightarrow \mathbb{K}$  is a *sesquilinear form* if for  $h, g$  in  $\mathcal{H}$ ,  $k, f$  in  $\mathcal{L}$ , and  $\alpha, \beta$  in  $\mathbb{K}$ ,

$$(i) \quad u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k);$$

$$(ii) \quad u(h, \alpha k + \beta f) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, f).$$

**Definition 2.3.12.** [4] A sesquilinear form  $u$  is *bounded* if there is a constant  $M$  such that  $|u(h, k)| \leq M \|h\| \|k\|$  for all  $h$  in  $\mathcal{H}$  and  $k$  in  $\mathcal{L}$ . The constant  $M$  is called a *bound* for  $u$ .

**Theorem 2.3.13.** [4] If  $u : \mathcal{H} \times \mathcal{L} \rightarrow \mathbb{K}$  is a bounded sesquilinear form with a bound  $M$ , then there are unique operators  $A$  in  $\mathcal{B}(\mathcal{H}, \mathcal{L})$  and  $B$  in  $\mathcal{B}(\mathcal{L}, \mathcal{H})$  such that

$$u(h, k) = \langle Ah, k \rangle = \langle h, Bk \rangle$$

for all  $h$  in  $\mathcal{H}$  and  $k$  in  $\mathcal{L}$  and both  $\|A\|$  and  $\|B\|$  are not greater than  $M$ .

**Definition 2.3.14.** [4] If  $A \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ , then the unique operator  $B$  in  $\mathcal{B}(\mathcal{L}, \mathcal{H})$  satisfying  $\langle Ah, k \rangle = \langle h, Bk \rangle$  is called the *adjoint* of  $A$  and is denoted by  $A^*$ .

**Proposition 2.3.15.** [4] If  $A, B \in \mathcal{B}(\mathcal{H})$ , where  $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$  and  $\alpha \in \mathbb{K}$ , then the following hold.

$$(1) \quad (\alpha A + B)^* = \overline{\alpha} A^* + B^*.$$

$$(2) \quad (AB)^* = B^* A^*.$$

$$(3) \quad A^{**} = (A^*)^* = A.$$

$$(4) \quad \text{If } A \text{ is invertible in } \mathcal{B}(\mathcal{H}), \text{ then } (A^*)^{-1} = (A^{-1})^*.$$

**Theorem 2.3.16.** [4] Let  $\mathcal{H}$  be an infinite dimensional separable Hilbert space with an orthonormal basis  $\{e_n\}$ . If  $T \in \mathcal{B}(\mathcal{H})$  with the matrix representation  $A = [a_{ji}]$  with respect to the basis  $\{e_n\}$ , then the matrix representation of  $T^*$  with respect to  $\{e_n\}$  is the matrix  $[\overline{a_{ji}}]^t$ .

**Definition 2.3.17.** [4] A bounded linear operator  $A$  on a Hilbert space  $\mathcal{H}$  is said to be *self-adjoint* if  $A = A^*$ .

**Theorem 2.3.18.** [4] If  $T \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ , the following statements are equivalent.

- (1)  $T$  is compact.
- (2)  $T^*$  is compact.
- (3) There is a sequence  $\{T_n\}$  of operators of finite rank such that  $\|T - T_n\| \rightarrow 0$ .

**Definition 2.3.19.** [4] If  $A \in \mathcal{B}(\mathcal{H})$ , a scalar  $\alpha$  is an *eigenvalue* of  $A$  if  $\ker(A - \alpha I) \neq \{0\}$ .

**Definition 2.3.20.** [4] If  $T \in \mathcal{B}(\mathcal{H})$ , then  $T$  is *positive* if  $\langle Th, h \rangle \geq 0$  for all  $h \in \mathcal{H}$ .

In symbols, this is denoted by  $T \geq 0$ . Note that every positive operator on a complex Hilbert space is self-adjoint.

**Theorem 2.3.21.** [4] If  $T$  is a positive compact operator on a Hilbert space  $\mathcal{H}$ , then there is a unique positive compact operator  $A$  such that  $A^2 = T$ .

**Definition 2.3.22.** [4] If  $T$  is a positive compact operator on a Hilbert space  $\mathcal{H}$ , then the unique positive compact operator  $A$  such that  $A^2 = T$  according to Theorem 2.3.21 is called the *positive square root* of  $T$  and denoted by  $|T|$ .

**Definition 2.3.23.** [4] A *partial isometry* is a linear operator  $W$  such that  $\|Wh\| = \|h\|$  for all  $h \in (\ker W)^\perp$ . The space  $(\ker W)^\perp$  is called the *initial space* of  $W$  and the space  $\text{ran } W$  is called the *final space* of  $W$ .

**Theorem 2.3.24.** [4] (Polar Decomposition) If  $T \in \mathcal{B}(\mathcal{H})$ , then there is a partial isometry  $W$  with  $(\ker T)^\perp$  as its initial space and  $\overline{\text{ran } T}$  as its final space such that  $T = W|T|$ . Moreover, if  $T = UP$  where  $P \geq 0$  and  $U$  is a partial isometry with  $\ker U = \ker P$ , then  $P = |T|$  and  $U = W$ .

**Theorem 2.3.25.** [4] (Spectral Theorem) If  $T$  is a compact self-adjoint operator on  $\mathcal{H}$ , then  $T$  has only a countable number of distinct eigenvalues. If  $\{\lambda_1, \lambda_2, \dots\}$  are the distinct nonzero eigenvalue of  $T$  with  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$ , and  $P_n$  is the projection of  $\mathcal{H}$  onto  $\ker(T - \lambda_n)$ , then  $P_n P_m = P_m P_n = 0$  if  $n \neq m$ , each  $\lambda_n$  is an real, and

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where the series converges to  $T$  in the metric defined by the norm of  $\mathcal{B}(\mathcal{H})$ .

**Corollary 2.3.26.** [4] *With the notation of Spectral Theorem. One has the following.*

- (1)  $\ker T = \overline{\text{span}(\bigcup_{n=1}^{\infty} P_n \mathcal{H})} = (\text{ran } T)^\perp$ ;
- (2) *each  $P_n$  has finite rank*;
- (3)  $\|T\| = \sup\{|\lambda_n| : n \geq 1\}$  and  $\lambda_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $T$  be a compact self-adjoint operator on  $\mathcal{H}$ . By Spectral Theorem,  $T$  has precisely finite or countable number of distinct eigenvalues. Let  $\{\lambda_n\}_{n=1}^{\infty}$  be the sequence of eigenvalues of  $T$  with  $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$ . For each  $n$ , let  $N_n$  be the dimension of  $\ker(T - \lambda_n)$ , and let  $\{\mu_n\}_{n=1}^{\infty} = \{\underbrace{\lambda_1, \dots, \lambda_1}_{N_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{N_2}, \dots\}$ . If  $T$  has only  $k$  eigenvalues, then we let  $\mu_n = 0$  for all for  $n > N_1 + N_2 + \dots + N_k$ .

**Corollary 2.3.27.** [4] *If  $T$  is a compact self-adjoint operator on  $\mathcal{H}$ , then there is an orthonormal basis  $\{e_n\}$  for  $(\ker T)^\perp$  such that*

$$Th = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n$$

for all  $h \in \mathcal{H}$ .

## 2.4 Schatten $p$ -Classes

Let  $\mathcal{H}$  be an infinite dimensional separable Hilbert space and  $K$  a compact operator on  $\mathcal{H}$ . Since  $0 \leq \|Kh\|^2 = \langle Kh, Kh \rangle = \langle K^*Kh, h \rangle$  for all  $h \in \mathcal{H}$ , it follows that  $K^*K$  is a positive compact operator. Whence, by Theorem 2.3.21, there is a unique positive compact operator  $|K|$  such that  $|K|^2 = K^*K$ . Since  $|K|$  is positive,  $|K|$  is self-adjoint. Thus, by Corollary 2.3.27, we have  $|K|h = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n$  for all  $h \in \mathcal{H}$ , where  $\{\mu_n\}_{n=1}^{\infty}$  is the sequence of eigenvalues of  $|K|$  and  $\{e_n\}_{n=1}^{\infty}$  is an orthonormal basis for  $(\ker |K|)^\perp$ . Notice that  $\ker K = \ker |K|$  due to the fact that  $\|Kh\|^2 = \langle Kh, Kh \rangle = \langle h, K^*Kh \rangle = \langle h, |K|^2h \rangle = \langle |K|h, |K|h \rangle = \||K|h\|^2$  for all  $h \in \mathcal{H}$ . We call the sequence  $\{\mu_n\}_{n=1}^{\infty}$  the *sequence of singular values* of the compact operator  $K$  and denote  $\mu_n$  by  $s_n(K)$  for all  $n$ . By Corollary 2.3.26, we have  $\|K\| = s_1(K) \geq s_2(K) \geq \dots \geq 0$  and  $\lim_{n \rightarrow \infty} s_n(K) = 0$ .

For  $1 \leq p \leq \infty$ , let

$$\mathcal{C}^p = \{K \in \mathcal{K}(\mathcal{H}) : \{s_k(K)\}_{k=1}^{\infty} \in l^p\}.$$

The set  $\mathcal{C}^p$  is called the *Schatten  $p$ -class*. We define, for  $1 \leq p < \infty$ , the norm  $\|\cdot\|_p$  on  $\mathcal{C}^p$  by

$$\|K\|_p = \left( \sum_{n=1}^{\infty} s_n(K)^p \right)^{1/p}.$$

For  $p = \infty$ , we define  $\|K\|_\infty = \sup_n s_n(K)$ . It is obvious that for any compact operator  $K$  on  $\mathcal{H}$ ,  $\|K\|_\infty = s_1(K) = \|K\|$ . Thus  $\mathcal{C}^\infty = \mathcal{K}(\mathcal{H})$ .

**Theorem 2.4.1.** [5] If  $K \in \mathcal{C}^1$ , then for each orthonormal basis  $\{e_n\}$  of  $\mathcal{H}$  the sum  $\sum_{n=1}^{\infty} \langle Ke_n, e_n \rangle$  is absolutely convergent and

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n \rangle = \sum_{n=1}^{\infty} s_n(K) \langle Ue_n, e_n \rangle,$$

where  $U$  is the unique partial isometry such that  $K = U|K|$ .

**Definition 2.4.2.** [5] For each  $K \in \mathcal{C}^1$ , the number

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n \rangle,$$

where  $\{e_n\}$  is an orthonormal basis of  $\mathcal{H}$ , is called the *trace* of  $K$ .

**Remark 2.4.3.** [5] If  $\{e_n\}$  is an ordered orthonormal basis of  $\mathcal{H}$  and  $K \in \mathcal{C}^1$  with the matrix representation  $A$  with respect to  $\{e_n\}$ , then the trace of  $K$  is exactly the sum of all entries in the main diagonal of  $A$ .

**Theorem 2.4.4.** [5] For each  $1 < p \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $(\mathcal{C}^p)^* \cong \mathcal{C}^q$ .

**Theorem 2.4.5.** [5]  $(\mathcal{C}^1)^* \cong \mathcal{B}(\mathcal{H})$ .

# Chapter 3

## Duality of Sequence Spaces of Infinite Matrices

### 3.1 Basic Results

Recall that, for any  $1 \leq r < \infty$ , the set  $\mathcal{L}^r$  of sequences of infinite matrices is defined by

$$\mathcal{L}^r = \left\{ \left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathcal{M}_{\infty} : \left[ \sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{B}(l^2) \right\}.$$

It is clear that if  $\{A_k\}_{k=1}^{\infty} \in \mathcal{L}^r$ , then  $A_k$  is necessarily a member of the absolute Schur algebra  $\mathcal{S}_{2,2}^r(\mathbb{C})$  for all  $k$ .

The following theorem was first stated in [3] by A. Charearnpol. It is a generalization of the characterization of the sequence spaces  $\mathcal{O}_b$  provided by J. Rakbud and S.-C. Ong in [11].

**Theorem 3.1.1.** *Let  $\left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$  be a sequence in  $\mathcal{B}(l^2)$  with  $a_{ji}^{(k)} \geq 0$  for all  $i, j, k$ .*

(1) *The sequence  $\left\{ \sum_{k=1}^n \left[ a_{ji}^{(k)} \right] \right\}_{n=1}^{\infty}$  is bounded in  $\mathcal{B}(l^2)$  if and only if  $\left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{B}(l^2)$ .*

(2) *If  $\left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{B}(l^2)$ , then  $\left\| \left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \right\| = \sup_n \left\| \sum_{k=1}^n \left[ a_{ji}^{(k)} \right] \right\|$ .*

From the above theorem, the following characterization of the set  $\mathcal{L}^r$  is immediately obtained.

**Corollary 3.1.2.** *Let  $\{A_k\}_{k=1}^{\infty}$  be a sequence in  $\mathcal{M}_{\infty}$  and  $1 \leq r < \infty$ . Then the following are equivalent:*

(1)  $\{A_k\}_{k=1}^{\infty}$  belongs to  $\mathcal{L}^r$ ;

(2)  $A_k \in \mathcal{S}_{2,2}^r(\mathbb{C})$  for all  $k$  and the sequence  $\left\{ \sum_{k=1}^n A_k^{[r]} \right\}_{n=1}^{\infty}$  is bounded in  $\mathcal{B}(l^2)$ ;

(3) the sequence  $\left\{ \left\| \sum_{k=1}^n A_k^{[r]} \right\| \right\}_{n=1}^{\infty}$  is bounded.

For any sequence  $\left\{ [a_{ji}^{(k)}] \right\}_{k=1}^{\infty}$  in  $\mathcal{M}_{\infty}$  and  $1 \leq r < \infty$ , we define

$$\left\| \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^{\infty} \right\|_r = \begin{cases} \left\| \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right] \right\|^{1/r} & \text{if } \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^{\infty} \in \mathcal{L}^r, \\ \infty & \text{otherwise.} \end{cases}$$

The following Hölder-type inequality was first established in [2] by Chaisuriya and Ong. It is useful for the research.

**Theorem 3.1.3.** (Hölder-type inequality) *For any  $A, B \in \mathcal{M}_{\infty}$  and  $1 < r < \infty$  with  $\frac{1}{r} + \frac{1}{r^*} = 1$ ,*

$$\| (A \bullet B)^{[1]} \| \leq \| A^{[r]} \|^{1/r} \| B^{[r^*]} \|^{1/r^*}$$

*under the conventions that  $\infty \cdot 0 = 0 \cdot \infty = 0$ ,  $\infty \cdot \alpha = \alpha \cdot \infty = \infty$  for all positive real number  $\alpha$  and  $\infty \cdot \infty = \infty$ .*

The Hölder and Minkowski-type inequalities below are extensions of the ones in [11].

**Theorem 3.1.4.** (Hölder-type inequality for sequences of matrices) *For any sequences  $\{A_k\}_{k=1}^{\infty}$  and  $\{B_k\}_{k=1}^{\infty}$  in  $\mathcal{M}_{\infty}$ ,*

$$\| \{A_k \bullet B_k\}_{k=1}^{\infty} \|_1 \leq \| \{A_k\}_{k=1}^{\infty} \|_r \| \{B_k\}_{k=1}^{\infty} \|_{r^*},$$

*where  $1 < r < \infty$  with  $\frac{1}{r} + \frac{1}{r^*} = 1$ , under the same convention as in Theorem 3.1.3.*

*Proof.* Let  $\left\{ A_k = [a_{ji}^{(k)}] \right\}_{k=1}^{\infty}$  and  $\left\{ B_k = [b_{ji}^{(k)}] \right\}_{k=1}^{\infty}$  be sequences in  $\mathcal{M}_{\infty}$ . If either  $\| \{A_k\}_{k=1}^{\infty} \|_r$  or  $\| \{B_k\}_{k=1}^{\infty} \|_{r^*}$  is  $\infty$ , then we are done. Suppose that both  $\| \{A_k\}_{k=1}^{\infty} \|_r$  and  $\| \{B_k\}_{k=1}^{\infty} \|_{r^*}$  are finite. Then  $\left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right]$  and  $\left[ \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^{r^*} \right]$  belong to  $\mathcal{B}(l^2)$ . Thus, by Hölder's inequality, we have for each  $i, j$  that

$$\sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \leq \left( \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right)^{1/r} \left( \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^{r^*} \right)^{1/r^*} < \infty.$$

Hence the matrix  $\left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \in \mathcal{M}_{\infty}$ . We want to show that  $\left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \in \mathcal{B}(l^2)$  and  $\| \| \{A_k \bullet B_k\}_{k=1}^{\infty} \| \|_1 \leq \| \| \{A_k\}_{k=1}^{\infty} \| \|_r \cdot \| \| \{B_k\}_{k=1}^{\infty} \| \|_{r^*}$ . By the Hölder-type inequality, we have

$$\begin{aligned} \left\| \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \right\| &\leq \left\| \left[ \left( \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right)^{1/r} \left( \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^{r^*} \right)^{1/r^*} \right] \right\| \\ &= \left\| \left[ \left( \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right)^{1/r} \right] \cdot \left[ \left( \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^{r^*} \right)^{1/r^*} \right] \right\| \\ &\leq \left\| \left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)}|^r \right] \right\|^{1/r} \left\| \left[ \sum_{k=1}^{\infty} |b_{ji}^{(k)}|^{r^*} \right] \right\|^{1/r^*} \\ &= \| \| \{A_k\}_{k=1}^{\infty} \| \|_r \cdot \| \| \{B_k\}_{k=1}^{\infty} \| \|_{r^*} < \infty. \end{aligned}$$

This implies that  $\left[ \sum_{k=1}^{\infty} |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \in \mathcal{B}(l^2)$ , which is equivalent to that  $\{A_k \bullet B_k\}_{k=1}^{\infty} \in \mathcal{L}^1$ , and  $\| \| \{A_k \bullet B_k\}_{k=1}^{\infty} \| \|_1 \leq \| \| \{A_k\}_{k=1}^{\infty} \| \|_r \cdot \| \| \{B_k\}_{k=1}^{\infty} \| \|_{r^*}$ .  $\square$

**Theorem 3.1.5.** (Minkowski-type inequality for sequences of matrices) *For any sequences  $\{A_k\}_{k=1}^{\infty}$  and  $\{B_k\}_{k=1}^{\infty}$  in  $\mathcal{M}_{\infty}$  and  $1 \leq r < \infty$ ,*

$$\| \| \{A_k + B_k\}_{k=1}^{\infty} \| \|_r \leq \| \| \{A_k\}_{k=1}^{\infty} \| \|_r + \| \| \{B_k\}_{k=1}^{\infty} \| \|_r$$

*under the conventions that  $\infty + \alpha = \alpha + \infty = \infty$  for all non-negative real number  $\alpha$  and  $\infty + \infty = \infty$ .*

*Proof.* For the case where either  $\| \| \{A_k\}_{k=1}^{\infty} \| \|_r = \infty$  or  $\| \| \{B_k\}_{k=1}^{\infty} \| \|_r = \infty$ , there is nothing to prove. Suppose that both  $\| \| \{A_k\}_{k=1}^{\infty} \| \|_r$  and  $\| \| \{B_k\}_{k=1}^{\infty} \| \|_r$  are finite. We assume first that  $1 < r < \infty$ . Then by the Hölder-type inequality for sequences of matrices, we have for each positive integer  $n$  that

$$\begin{aligned} \left\| \sum_{k=1}^n (A_k + B_k)^{[r]} \right\| &= \left\| \sum_{k=1}^n (A_k + B_k)^{[1]} \bullet (A_k + B_k)^{[r-1]} \right\| \\ &\leq \left\| \sum_{k=1}^n \left( A_k^{[1]} + B_k^{[1]} \right) \bullet (A_k + B_k)^{[r-1]} \right\| \\ &= \left\| \sum_{k=1}^n A_k^{[1]} \bullet (A_k + B_k)^{[r-1]} + \sum_{k=1}^n B_k^{[1]} \bullet (A_k + B_k)^{[r-1]} \right\| \\ &\leq \left\| \sum_{k=1}^n A_k^{[1]} \bullet (A_k + B_k)^{[r-1]} \right\| + \left\| \sum_{k=1}^n B_k^{[1]} \bullet (A_k + B_k)^{[r-1]} \right\| \end{aligned}$$



$$\begin{aligned}
&\leq \left\| \sum_{k=1}^n A_k^{[r]} \right\|^{1/r} \left\| \sum_{k=1}^n ((A_k + B_k)^{[r-1]})^{[r^*]} \right\|^{1/r^*} \\
&\quad + \left\| \sum_{k=1}^n B_k^{[r]} \right\|^{1/r} \left\| \sum_{k=1}^n ((A_k + B_k)^{[r-1]})^{[r^*]} \right\|^{1/r^*} \\
&= \left( \left\| \sum_{k=1}^n A_k^{[r]} \right\|^{1/r} + \left\| \sum_{k=1}^n B_k^{[r]} \right\|^{1/r} \right) \left\| \sum_{k=1}^n (A_k + B_k)^{[r]} \right\|^{1/r^*},
\end{aligned}$$

where  $\frac{1}{r} + \frac{1}{r^*} = 1$ , which implies that

$$\begin{aligned}
\left\| \sum_{k=1}^n (A_k + B_k)^{[r]} \right\|^{1/r} &\leq \left\| \sum_{k=1}^n A_k^{[r]} \right\|^{1/r} + \left\| \sum_{k=1}^n B_k^{[r]} \right\|^{1/r} \\
&\leq \|\{A_k\}_{k=1}^\infty\|_r + \|\{B_k\}_{k=1}^\infty\|_r.
\end{aligned}$$

For the case where  $r = 1$ , we easily have for each positive integer  $n$  that

$$\begin{aligned}
\left\| \sum_{k=1}^n (A_k + B_k)^{[1]} \right\| &\leq \left\| \sum_{k=1}^n A_k^{[1]} + \sum_{k=1}^n B_k^{[1]} \right\| \\
&\leq \left\| \sum_{k=1}^n A_k^{[1]} \right\| + \left\| \sum_{k=1}^n B_k^{[1]} \right\| \\
&\leq \|\{A_k\}_{k=1}^\infty\|_1 + \|\{B_k\}_{k=1}^\infty\|_1
\end{aligned}$$

as well. Thus, by Corollary 3.1.2, the sequence  $\{A_k + B_k\}_{k=1}^\infty$  belongs to  $\mathcal{L}^r$  and

$$\|\{A_k + B_k\}_{k=1}^\infty\|_r \leq \|\{A_k\}_{k=1}^\infty\|_r + \|\{B_k\}_{k=1}^\infty\|_r$$

for all  $1 \leq r < \infty$ . The proof is complete.  $\square$

The following lemma was first stated and proved in [10]. It is a beautiful consequence of the Hölder-type inequality.

**Lemma 3.1.6.** *For any  $1 \leq r < \infty$  and matrices  $A$  and  $B$  in  $S_{2,2}^r(\mathbb{C})$ ,*

$$\|A^{[r]} - B^{[r]}\| \leq (\|A\|_{2,2,r} + \|B\|_{2,2,r}) \|A - B\|_{2,2,r}.$$

The proposition below was first stated and proved in [10] as well. We can see that it follows easily from the lemma above.

**Proposition 3.1.7.** *For any  $1 \leq r < \infty$ , the map  $A \mapsto A^{[r]}$  from  $S_{2,2}^r(\mathbb{C})$  into  $\mathcal{B}(l^2)$  is continuous.*

**Theorem 3.1.8.** *For each  $1 \leq r < \infty$ , the set  $\mathcal{L}^r$  equipped with the norm  $\|\cdot\|_r$  is a Banach space.*

*Proof.* From Minkowski's inequality for sequences of matrices, we have that the set  $\mathcal{L}^r$  endowed with the norm  $\|\cdot\|_r$  is a normed space. To see that it is a Banach space, let  $\{\mathbf{A}_n = \{A_k^{(n)}\}_{k=1}^\infty\}_{n=1}^\infty$  be a Cauchy sequence in  $\mathcal{L}^r$ . As we have for each  $k$  that

$$\|A_k^{(n)} - A_k^{(m)}\|_{2,2,r} \leq \|\mathbf{A}_n - \mathbf{A}_m\|_r \quad \text{for all } n, m,$$

it follows that the sequence  $\{A_k^{(n)}\}_{n=1}^\infty$  is a Cauchy sequence in  $\mathcal{S}_{2,2}^r(\mathbb{C})$  for all  $k$ . So, for each  $k$ , we obtain by the completeness of  $\mathcal{S}_{2,2}^r(\mathbb{C})$  that there exists an  $A_k$  in  $\mathcal{S}_{2,2}^r(\mathbb{C})$  such that  $A_k^{(n)} \rightarrow A_k$ . Let  $\mathbf{A} = \{A_k\}_{k=1}^\infty$ . We claim that  $\mathbf{A} \in \mathcal{L}^r$  and  $\mathbf{A}_n \rightarrow \mathbf{A}$ . To prove these, let  $\epsilon > 0$  be given. Then there is a positive integer  $N$  such that for each positive integer  $K$ ,

$$\left\| \sum_{k=1}^K (A_k^{(n)} - A_k^{(m)})^{[r]} \right\|^{1/r} \leq \|\mathbf{A}_n - \mathbf{A}_m\|_r < \frac{\epsilon}{2} \quad \text{for all } n, m \geq N. \quad (*)$$

Since  $A_k^{(n)} \rightarrow A_k$  in  $\mathcal{S}_{2,2}^r(\mathbb{C})$  for all  $k$ , it follows for each fixed  $n$  that  $A_k^{(n)} - A_k^{(m)} \rightarrow A_k^{(n)} - A_k$  in  $\mathcal{S}_{2,2}^r(\mathbb{C})$  for all  $k$ . Thus, by Proposition 3.1.7, we obtain for each fixed  $n$  that  $(A_k^{(n)} - A_k^{(m)})^{[r]} \rightarrow (A_k^{(n)} - A_k)^{[r]}$  in  $\mathcal{B}(l^2)$  for all  $k$ . From this we have for each fixed  $n$  and  $K$  that  $\sum_{k=1}^K (A_k^{(n)} - A_k^{(m)})^{[r]} \rightarrow \sum_{k=1}^K (A_k^{(n)} - A_k)^{[r]}$  in  $\mathcal{B}(l^2)$ . Whence, by taking the limits as  $m \rightarrow \infty$  on both sides of (\*), we obtain by the continuity of the operator norm on  $\mathcal{B}(l^2)$  that for each  $n \geq N$ ,

$$\left\| \sum_{k=1}^K (A_k^{(n)} - A_k)^{[r]} \right\|^{1/r} \leq \frac{\epsilon}{2} \quad \text{for all } K \geq 1.$$

Therefore, by Theorem 3.1.1,

$$\|\mathbf{A}_n - \mathbf{A}\|_r = \sup_K \left\| \sum_{k=1}^K (A_k^{(n)} - A_k)^{[r]} \right\|^{1/r} < \epsilon \quad \text{for all } n \geq N. \quad (**)$$

The inequality (\*\*) yields that  $\mathbf{A}_N - \mathbf{A}$  belongs to  $\mathcal{L}^r$ , which implies that  $\mathbf{A} = \mathbf{A}_N - (\mathbf{A}_N - \mathbf{A})$  is an element of  $\mathcal{L}^r$ . Consequently, by (\*\*) again, we get  $\mathbf{A}_n \rightarrow \mathbf{A}$ .  $\square$

## 3.2 Duality

In this section, we study the duality of the sequence spaces  $\mathcal{L}^r$ . The aim is to decompose the dual space  $(\mathcal{L}^r)^*$  of  $\mathcal{L}^r$  as an  $l^1$  direct-sum of its two closed subspaces. Before getting the results, we need some notational conventions.

For any  $z \in \mathbb{C}$ , we define the function  $\text{sgn}(\cdot)$  on  $\mathbb{C}$  by

$$\text{sgn}(z) = \begin{cases} \frac{\bar{z}}{|z|} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

For any sequences  $\mathbf{A} = \{A_k\}_{k=1}^\infty$  and  $\mathbf{B} = \{B_k\}_{k=1}^\infty$  in  $\mathcal{M}_\infty$  and any positive integer  $n$ , we let  $\mathbf{A} \bullet \mathbf{B} = \{A_k \bullet B_k\}_{k=1}^\infty$ ,  $\mathbf{A}_{n_\downarrow} = \{(A_k)_{n_\downarrow}\}_{k=1}^\infty$ ,  $\mathbf{A}_{n_r} = \{(A_k)_{n_r}\}_{k=1}^\infty$ , and  $\mathbf{A}_{n\downarrow} = \{A_1, A_2, \dots, A_n, 0, 0, \dots\}$ . It is clear that  $(\mathbf{A}_{K\downarrow})_{n_\downarrow} = (\mathbf{A}_{n_\downarrow})_{K\downarrow}$  for all positive integers  $n$  and  $K$ . Notice that for each  $1 \leq r < \infty$ , if  $\mathbf{A} = \left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^\infty \in \mathcal{L}^r$ , then each of the following holds true:

$$(i) \quad \|\mathbf{A}_{n_\downarrow}\|_r = \left\| \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right]_{n_\downarrow} \right\|^{1/r},$$

$$(ii) \quad \|\mathbf{A}_{K\downarrow}\|_r = \left\| \sum_{k=1}^K A_k^{[r]} \right\|^{1/r} = \left\| \left[ \sum_{k=1}^K |a_{ji}^{(k)}|^r \right] \right\|^{1/r} \quad \text{and}$$

$$(iii) \quad \|\mathbf{A}_{n_\downarrow} - \mathbf{A}\|_r = \|\{(A_k)_{n_\downarrow} - A_k\}_{k=1}^\infty\|_r = \left\| \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right]_{n_\downarrow} - \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right] \right\|^{1/r},$$

for all  $n$  and  $K$ . The first and second equations imply that  $\|\mathbf{A}\|_r = \sup_n \|\mathbf{A}_{n_\downarrow}\|_r$

and  $\|\mathbf{A}\|_r = \sup_K \|\mathbf{A}_{K\downarrow}\|_r$  respectively. And the last one implies that the matrix

$\left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right]$  is compact if and only if  $\|\mathbf{A}_{n_\downarrow} - \mathbf{A}\|_r \rightarrow 0$ . For each  $A = [a_{ji}]$

in  $\mathcal{M}_\infty$  and positive integer  $k$ , let  $\sum A = \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ji}$  if the series converges, let

$\text{sgn}A = [\text{sgn}(a_{ji})]$ , and let  $s(A; k)$  be the sequence whose  $k$ -th term is the matrix  $A$  and all other terms are 0. Finally, for any  $\lambda \in \mathbb{C}$  and pair  $(j, i)$  of positive integers, let  $E(\lambda; (j, i))$  be the matrix whose  $(j, i)$ -th entry is the number  $\lambda$  and all other entries are 0.

On the classical sequence spaces  $l^p$ , there is a result closely related to their duality as follows: for  $1 \leq p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , a sequence  $\mathbf{x}$  belongs to  $l^q$  if and only if  $\mathbf{x}$  ‘‘Schur multiplies’’ every  $\mathbf{y}$  in  $l^p$  into  $l^1$ . An analogue of this result is also obtained for our sequence spaces  $\mathcal{L}^r$  of infinite matrices.

**Theorem 3.2.1.** *Let  $1 < r < \infty$  with  $\frac{1}{r} + \frac{1}{r^*} = 1$ .*

(1)  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^{r^*}$  if and only if  $\{A_k \bullet B_k\}_{k=1}^\infty \in \mathcal{L}^1$  for all  $\{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ .

(2) If  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^{r^*}$ , then

$$\|\{A_k\}_{k=1}^\infty\|_{r^*} = \sup\{\|\{A_k \bullet B_k\}_{k=1}^\infty\|_1 : \{B_k\}_{k=1}^\infty \in \mathcal{L}^r, \|\{B_k\}_{k=1}^\infty\|_r \leq 1\}.$$

*Proof.* (1). Let  $\mathbf{A} = \left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^\infty$  be a sequence in  $\mathcal{M}_\infty$ . Suppose that  $\{A_k \bullet B_k\}_{k=1}^\infty \in \mathcal{L}^1$  for all  $\{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ . We want to show that  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^{r^*}$ . By the assumption, a map  $\Psi : \mathcal{L}^r \rightarrow \mathcal{L}^1$  can be defined as follows:  $\Psi(\{B_k\}_{k=1}^\infty) = \{A_k \bullet B_k\}_{k=1}^\infty$  for all  $\{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ . For any positive integer  $n$ , let  $\Psi_n : \mathcal{L}^r \rightarrow \mathcal{L}^1$  be

defined by  $\Psi_n(\{B_k\}_{k=1}^\infty) = \mathbf{A}_n \bullet \mathbf{B}$  for all  $\mathbf{B} = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ . Then by the Hölder-type inequality for sequences of matrices, we have for every  $\mathbf{B} = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$  that

$$\|\Psi_n(\{B_k\}_{k=1}^\infty)\|_1 = \|\mathbf{A}_n \bullet \mathbf{B}\|_1 \leq \|\mathbf{A}_n\|_{r^*} \|\mathbf{B}\|_r.$$

So the operator  $\Psi_n$  is bounded for all  $n$ . For each  $\{B_k = [b_{ji}^{(k)}]\}_{k=1}^\infty \in \mathcal{L}^r$ , we have

$$\begin{aligned} \|\Psi_n(\{B_k\}_{k=1}^\infty)\|_1 &= \left\| \left[ \sum_{k=1}^n |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \right\| \leq \left\| \left[ \sum_{k=1}^\infty |a_{ji}^{(k)} b_{ji}^{(k)}| \right] \right\| \\ &= \|\{A_k \bullet B_k\}_{k=1}^\infty\|_1 \text{ for all } n. \end{aligned}$$

Hence, by the uniform boundedness principle, the set  $\{\|\Psi_n\| : n = 1, 2, 3, \dots\}$  is bounded. For every  $\mathbf{B} = \{B_k\}_{k=1}^\infty \in \mathcal{L}^r$  with  $\|\mathbf{B}\|_r \leq 1$ , we have by Theorem 3.1.1 that

$$\begin{aligned} \|\Psi(\mathbf{B})\|_1 &= \sup_n \left\| \sum_{k=1}^n (A_k \bullet B_k)^{[1]} \right\| = \sup_n \|\mathbf{A}_n \bullet \mathbf{B}\|_1 \\ &= \sup_n \|\Psi_n(\mathbf{B})\|_1 \leq \sup_n \|\Psi_n\|. \end{aligned}$$

Thus, by the boundedness of the set  $\{\|\Psi_n\| : n = 1, 2, 3, \dots\}$ , the operator  $\Psi$  is bounded. Next, let  $\mathbf{D} = \{A_k^{[r^*-1]}\}_{k=1}^\infty$ . Then  $(\mathbf{D}_{n_\perp})_K \in \mathcal{L}^r$  for all  $n, K$ . Thus

$$\begin{aligned} \left\| \left( \sum_{k=1}^K A_k^{[r^*]} \right)_{n_\perp} \right\| &= \left\| \sum_{k=1}^K (A_k^{[r^*]})_{n_\perp} \right\| = \left\| \sum_{k=1}^K A_k^{[1]} \bullet (A_k^{[r^*-1]})_{n_\perp} \right\| \\ &= \|\mathbf{A} \bullet (\mathbf{D}_{n_\perp})_K\|_1 = \|\Psi((\mathbf{D}_{n_\perp})_K)\|_1 \\ &\leq \|\Psi\| \|(\mathbf{D}_{n_\perp})_K\|_r = \|\Psi\| \left\| \left( \sum_{k=1}^K (A_k^{[r^*-1]})^{[r]} \right)_{n_\perp} \right\|^{1/r} \\ &= \|\Psi\| \left\| \left( \sum_{k=1}^K A_k^{[r^*]} \right)_{n_\perp} \right\|^{1/r} \text{ for all } n, K. \end{aligned} \quad (\star)$$

It follows that

$$\left\| \left( \sum_{k=1}^K A_k^{[r^*]} \right)_{n_\perp} \right\| \leq \|\Psi\|^{r^*} \text{ for all } n, K.$$

Whence, by Theorem 1.1(2), we obtain for each  $K$  that  $\sum_{k=1}^K A_k^{[r^*]} \in \mathcal{B}(l^2)$  and by Theorem 1.1(3),

$$\left\| \sum_{k=1}^K A_k^{[r^*]} \right\| = \sup_n \left\| \left( \sum_{k=1}^K A_k^{[r^*]} \right)_{n_\perp} \right\| \leq \|\Psi\|^{r^*}.$$

Therefore, by Corollary 3.1.2, the sequence  $\mathbf{A}$  belongs to  $\mathcal{L}^{r^*}$ . Conversely, suppose that  $\mathbf{A} \in \mathcal{L}^{r^*}$ . Then for any  $\{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ , we have by the Hölder-type inequality for sequences of matrices that  $\{A_k \bullet B_k\}_{k=1}^\infty \in \mathcal{L}^1$ .

(2). Suppose that  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^{r^*}$ . Then by (1), the linear operator  $\Psi : \mathcal{L}^r \rightarrow \mathcal{L}^1$  defined by  $\{B_k\}_{k=1}^\infty \mapsto \{A_k \bullet B_k\}_{k=1}^\infty$  is well-defined, and by the Hölder-type inequality for sequences of matrices, it is obvious that  $\|\Psi\| \leq \|\{A_k\}_{k=1}^\infty\|_{r^*}$ . By the same argument as given in the proof of (1) (see the argument to obtain the inequality  $(\star)$ ), we have

$$\left\| \sum_{k=1}^n A_k^{[r^*]} \right\|^{1/r^*} \leq \|\Psi\| \text{ for all } n.$$

It follows from Theorem 3.1.1 that  $\|\{A_k\}_{k=1}^\infty\|_{r^*} \leq \|\Psi\|$ . Consequently, we obtain

$$\|\{A_k\}_{k=1}^\infty\|_{r^*} = \|\Psi\| = \sup\{\|\{A_k \bullet B_k\}_{k=1}^\infty\|_1 : \{B_k\}_{k=1}^\infty \in \mathcal{L}^r, \|\{B_k\}_{k=1}^\infty\|_r \leq 1\}$$

as required. The proof is complete.  $\square$

For each  $1 \leq r < \infty$ , let

$$\mathcal{L}_\kappa^r = \left\{ \left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^\infty \in \mathcal{M}_\infty : \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right] \in \mathcal{K}(l^2) \right\}.$$

The following results on the sets  $\mathcal{L}_\kappa^r$  are evident.

- (i)  $\mathcal{L}_\kappa^r \subsetneq \mathcal{L}^r$ .
- (ii) A sequence  $\mathbf{A}$  in  $\mathcal{M}_\infty$  belongs to  $\mathcal{L}_\kappa^r$  if and only if  $\|\mathbf{A} - \mathbf{A}_{n_\downarrow}\|_r \rightarrow 0$ .
- (iii) If a sequence  $\mathbf{A}$  belongs to  $\mathcal{L}_\kappa^r$ , then  $\mathbf{A} - \mathbf{A}_{n_\downarrow}$  belongs to  $\mathcal{L}_\kappa^r$  for all  $n$ .

The following theorem is a more general version of the characterization of the sequence space  $\mathcal{O}_\kappa$  provided by Rakbud et al. in [11].

**Theorem 3.2.2.** *Let  $\{A_k = [a_{ji}^{(k)}]\}_{k=1}^\infty$  be a sequence in  $\mathcal{M}_\infty$  with  $a_{jk}^{(k)} \geq 0$  for all  $i, j, k$ . Then  $\left[ \sum_{k=1}^\infty a_{ji}^{(k)} \right] \in \mathcal{K}(l^2)$  if and only if  $A_k \in \mathcal{K}(l^2)$  for all  $k$  and the sequence  $\left\{ \sum_{k=1}^n A_k \right\}_{k=1}^\infty$  converges in  $\mathcal{B}(l^2)$ .*

*Proof.* Suppose that the matrix  $A = \left[ \sum_{k=1}^\infty a_{ji}^{(k)} \right]$  is compact. Then for each  $k$ , we have by Theorem 1.1(1) that  $A_k \in \mathcal{B}(l^2)$  and

$$\|A_k - (A_k)_{n_\downarrow}\| \leq \|A - A_{n_\downarrow}\| \rightarrow 0.$$

Thus  $A_k$  is compact for all  $k$ . To see that the sequence  $\left\{ \sum_{k=1}^n A_k \right\}_{k=1}^\infty$  converges in  $\mathcal{B}(l^2)$ , let  $\epsilon > 0$  be given. Then by the compactness of the matrix  $A$ , there exists

a positive integer  $N$  such that  $\|A_{N_j} - A\| < \frac{\epsilon}{3}$ . As the series  $\sum_{k=1}^{\infty} a_{ji}^{(k)}$  converges for all  $1 \leq j, i \leq N$ , there is a positive integer  $K_0$  such that for each  $1 \leq j, i \leq N$ ,  $\sum_{k=K}^{\infty} a_{ji}^{(k)} < \frac{\epsilon}{3N^{3/2}}$  for all  $K \geq K_0$ . Hence for each  $K \geq K_0$ ,

$$\begin{aligned} \left\| \sum_{k=1}^K A_k - A \right\| &\leq \left\| A_{N_j} - \left( \sum_{k=1}^K A_k \right)_{N_j} \right\| + \left\| \left( \sum_{k=1}^K A_k \right)_{N_j} - \sum_{k=1}^K A_k \right\| \\ &\quad + \|A_{N_j} - A\| \\ &\leq \left\{ \sum_{j=1}^N \left( \sum_{i=1}^N \sum_{k=K}^{\infty} a_{ji}^{(k)} \right)^2 \right\}^{1/2} + 2 \|A_{N_j} - A\| \\ &< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon. \end{aligned}$$

This yields  $\sum_{k=1}^{\infty} A_k = A$  in  $\mathcal{B}(l^2)$ . Conversely, suppose that  $A_k$  is compact for all  $k$  and

that the sequence  $\left\{ \sum_{k=1}^n A_k \right\}_{n=1}^{\infty}$  converges in  $\mathcal{B}(l^2)$ . It is clear that  $\sum_{k=1}^{\infty} A_k = \left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right]$ .

Since  $\mathcal{K}(l^2)$  is closed in  $\mathcal{B}(l^2)$ , it follows that  $\sum_{k=1}^{\infty} A_k$  is compact. Thus we obtain that

$\left[ \sum_{k=1}^{\infty} a_{ji}^{(k)} \right]$  is compact as required.  $\square$

The following characterization of the set  $\mathcal{L}_{\kappa}^r$  is an immediate consequence of Theorem 3.2.2 above.

**Corollary 3.2.3.** *Let  $\{A_k\}_{k=1}^{\infty}$  be a sequence in  $\mathcal{M}_{\infty}$  and  $1 \leq r < \infty$ . Then  $\{A_k\}_{k=1}^{\infty} \in \mathcal{L}_{\kappa}^r$  if and only if  $A_k^{[r]}$  is compact for all  $k$  and the sequence  $\left\{ \sum_{k=1}^n A_k^{[r]} \right\}_{k=1}^{\infty}$  converges in  $\mathcal{B}(l^2)$ .*

**Theorem 3.2.4.** *For each  $1 \leq r < \infty$ , the set  $\mathcal{L}_{\kappa}^r$  is a Banach subspace of  $\mathcal{L}^r$ .*

*Proof.* For any matrix  $A \in \mathcal{M}_{\infty}$  and positive integer  $n$ , we let here for convenience  $A_{\lrcorner n} = A - A_{n-1}$ . We will show first that  $\mathcal{L}_{\kappa}^r$  is a normed subspace of  $\mathcal{L}^r$ . Let  $\{A_k\}_{k=1}^{\infty}, \{B_k\}_{k=1}^{\infty} \in \mathcal{L}_{\kappa}^r$ . Then

$$\begin{aligned} \|\{(A_k + B_k)_{\lrcorner n}\}_{k=1}^{\infty}\|_r &= \|\{(A_k)_{\lrcorner n}\}_{k=1}^{\infty} + \{(B_k)_{\lrcorner n}\}_{k=1}^{\infty}\|_r \\ &\leq \|\{(A_k)_{\lrcorner n}\}_{k=1}^{\infty}\|_r + \|\{(B_k)_{\lrcorner n}\}_{k=1}^{\infty}\|_r \rightarrow 0. \end{aligned}$$

Thus  $\mathcal{L}_{\kappa}^r$  is closed under addition. It clear that  $\lambda\{A_k\}_{k=1}^{\infty} \in \mathcal{L}_{\kappa}^r$  for any complex number  $\lambda$ . Hence  $\mathcal{L}_{\kappa}^r$  is a normed subspace of  $\mathcal{L}^r$ . To show that  $\mathcal{L}_{\kappa}^r$  is a Banach space, it suffices to show that  $\mathcal{L}_{\kappa}^r$  is a closed subspace of  $\mathcal{L}^r$ . Suppose that

$\left\{ \mathbf{A}_n = \left\{ A_k^{(n)} \right\}_{k=1}^\infty \right\}_{n=1}^\infty$  is a sequence in  $\mathcal{L}_\kappa^r$  converging to an element  $\mathbf{A} = \{A_k\}_{k=1}^\infty$  in  $\mathcal{L}^r$ , and let  $\epsilon > 0$  be given. Then there is a positive integer  $N$  such that

$$\|\mathbf{A}_N - \mathbf{A}\|_r < \frac{\epsilon}{2}.$$

Due to the membership of  $\mathbf{A}_N$  in  $\mathcal{L}_\kappa^r$ , we have that there exists a positive integer  $J_0$  such that

$$\left\| \left\{ \left( A_k^{(N)} \right)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r < \frac{\epsilon}{2} \text{ for all } J \geq J_0.$$

It follows that

$$\begin{aligned} \left\| \left\{ (A_k)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r &\leq \left\| \left\{ \left( A_k^{(N)} \right)_{\downarrow J} \right\}_{k=1}^\infty - \left\{ (A_k)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r + \left\| \left\{ \left( A_k^{(N)} \right)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r \\ &= \left\| \left\{ \left( A_k^{(N)} - A_k \right)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r + \left\| \left\{ \left( A_k^{(N)} \right)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r \\ &\leq \|\mathbf{A}_N - \mathbf{A}\|_r + \left\| \left\{ \left( A_k^{(N)} \right)_{\downarrow J} \right\}_{k=1}^\infty \right\|_r \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } J \geq J_0. \end{aligned}$$

Consequently,  $\{A_k\}_{k=1}^\infty$  belongs to  $\mathcal{L}_\kappa^r$ . □

The following is the main theorem in this thesis. It tells us that the annihilator  $(\mathcal{L}_\kappa^r)^\perp$  of  $\mathcal{L}_\kappa^r$  is complemented in  $(\mathcal{L}^r)^*$ . Furthermore, the norm of the decomposition of any bounded linear functional on  $\mathcal{L}^r$  is additive.

**Theorem 3.2.5.** *Let  $1 \leq r < \infty$ . Then the following hold.*

- (1) *The annihilator  $(\mathcal{L}_\kappa^r)^\perp$  of  $\mathcal{L}_\kappa^r$  is a non-trivial closed subspace of the dual  $(\mathcal{L}^r)^*$  of  $\mathcal{L}^r$ .*
- (2) *There is a subspace  $\mathcal{P}$  of  $(\mathcal{L}^r)^*$  such that  $\mathcal{P}$  is isometrically isomorphic to  $(\mathcal{L}_\kappa^r)^*$  and  $(\mathcal{L}^r)^* = \mathcal{P} \oplus (\mathcal{L}_\kappa^r)^\perp$ .*
- (3) *For any  $f \in (\mathcal{L}^r)^*$ , the decomposition  $f = g + h$ , where  $g \in \mathcal{P}$  and  $h \in (\mathcal{L}_\kappa^r)^\perp$ , satisfies  $\|f\| = \|g\| + \|h\|$ .*

*Proof.* (1). By the Hahn-Banach extension theorem, we have in general that if  $A$  is a non-trivial closed subspace of a Banach space  $X$ , then the annihilator  $A^\perp$  of  $A$  is a non-trivial closed subspace of the dual  $X^*$  of  $X$ . Thus, by this fact, the assertion (1) holds.

(2). Let  $\varphi \in (\mathcal{L}^r)^*$ . For each  $k$ , let  $\varphi_k : \mathcal{S}_{2,2}^r(\mathbb{C}) \rightarrow \mathbb{C}$  be defined by  $\varphi_k(A) = \varphi(s(A; k))$  for all  $A \in \mathcal{S}_{2,2}^r(\mathbb{C})$ . It is easy to see that  $\varphi_k$  is linear and  $\|\varphi_k\| \leq \|\varphi\|$  for all  $k$ . Hence, for each  $k$ , the map  $\varphi_k$  belongs to  $(\mathcal{S}_{2,2}^r(\mathbb{C}))^*$ . Next, let  $B_k^{(\varphi)} = [\varphi_k(E(1; (j, i)))]$  for all  $k$ , and let  $\mathbf{B}^{(\varphi)} = \left\{ B_k^{(\varphi)} \right\}_{k=1}^\infty$ . We want to show first that  $\sum_{k=1}^\infty \sum \left( A_k \bullet B_k^{(\varphi)} \right)^{[1]} < \infty$  for all  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^r$ . To see this,

let  $\mathbf{A} = \left\{ A_k = \left[ a_{ji}^{(k)} \right]_{k=1}^\infty \right\} \in \mathcal{L}^r$ . Notice that  $\mathbf{A} \bullet \mathbf{B}^{(\varphi)}$  belongs to  $\mathcal{L}^r$  due to the fact that  $\sum_{k=1}^\infty \left| a_{ji}^{(k)} \varphi_k(E(1; (j, i))) \right|^r \leq \|\varphi\|^r \sum_{k=1}^\infty \left| a_{ji}^{(k)} \right|^r$  for all  $j, i$ . For each  $k$ , let  $\widetilde{A}_k = \left[ \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) a_{ji}^{(k)} \right]$ , and let  $\widetilde{\mathbf{A}} = \left\{ \widetilde{A}_k \right\}_{k=1}^\infty$ . Then  $\widetilde{\mathbf{A}} \in \mathcal{L}^r$  with the same norm as  $\mathbf{A}$ . Let  $\nu, \mu$  and  $K$  be positive integers, and let  $n = \max\{\nu, \mu\}$ . Then

$$\begin{aligned}
\sum_{k=1}^K \sum_{j=1}^\nu \sum_{i=1}^\mu \left| a_{ji}^{(k)} \varphi_k(E(1; (j, i))) \right| &< \sum_{k=1}^K \sum_{j=1}^n \sum_{i=1}^n \left| \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right| \\
&= \sum_{k=1}^K \sum_{j=1}^n \sum_{i=1}^n \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \\
&= \sum_{k=1}^K \sum_{j=1}^n \sum_{i=1}^n \varphi_k \left( E \left( \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) a_{ji}^{(k)}; (j, i) \right) \right) \\
&= \sum_{k=1}^K \varphi_k \left( \sum_{j=1}^n \sum_{i=1}^n E \left( \left( \text{sgn} \left( \varphi_k \left( E \left( a_{ji}^{(k)}; (j, i) \right) \right) \right) \right) a_{ji}^{(k)}; (j, i) \right) \right) \\
&= \sum_{k=1}^K \varphi_k \left( \left( \widetilde{A}_k \right)_{n \times n} \right) = \sum_{k=1}^K \varphi \left( s \left( \left( \widetilde{A}_k \right)_{n \times n}; k \right) \right) \\
&= \varphi \left( \sum_{k=1}^K s \left( \left( \widetilde{A}_k \right)_{n \times n}; k \right) \right) = \varphi \left( \left( \widetilde{\mathbf{A}}_{n \times n} \right)_{K \times 1} \right) \\
&\leq \|\varphi\| \left\| \left( \widetilde{\mathbf{A}}_{n \times n} \right)_{K \times 1} \right\|_r \leq \|\varphi\| \left\| \widetilde{\mathbf{A}} \right\|_r = \|\varphi\| \left\| \mathbf{A} \right\|_r.
\end{aligned}$$

It follows that

$$\sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{i=1}^\infty \left| a_{ji}^{(k)} \varphi_k(E(1; (j, i))) \right| \leq \|\varphi\| \left\| \mathbf{A} \right\|_r.$$

From this result, we can define a bounded linear functional  $\psi_\varphi$  on  $\mathcal{L}^r$  by  $\{A_k\}_{k=1}^\infty \mapsto \sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$  with  $\|\psi_\varphi\| \leq \|\varphi\|$ . Notice that for any  $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathcal{L}^r$  and positive integer  $K$ , we have by the absolute convergence of the series  $\sum A_k \bullet B_k^{(\varphi)}$  ( $k = 1, 2, \dots, K$ ) that

$$\begin{aligned}
\psi_\varphi(\mathbf{A}_{K \times 1}) &= \sum_{k=1}^K \sum A_k \bullet B_k^{(\varphi)} = \sum_{k=1}^K \lim_{n \rightarrow \infty} \sum \left( A_k \bullet B_k^{(\varphi)} \right)_{n \times 1} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^K \sum \left( A_k \bullet B_k^{(\varphi)} \right)_{n \times 1} = \lim_{n \rightarrow \infty} \sum_{k=1}^K \varphi_k \left( (A_k)_{n \times 1} \right) \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^K \varphi \left( s \left( (A_k)_{n \times 1}; k \right) \right) = \lim_{n \rightarrow \infty} \varphi \left( \sum_{k=1}^K s \left( (A_k)_{n \times 1}; k \right) \right) \\
&= \lim_{n \rightarrow \infty} \varphi \left( (\mathbf{A}_{n \times 1})_{K \times 1} \right) = \lim_{n \rightarrow \infty} \varphi \left( (\mathbf{A}_{K \times 1})_{n \times 1} \right). \tag{§}
\end{aligned}$$



Next, let  $\rho_\varphi = \varphi - \psi_\varphi$ . We will show that  $\rho_\varphi \in (\mathcal{L}_\kappa^r)^\perp$ . To see this, let  $\mathbf{A} \in \mathcal{L}_\kappa^r$ . Then  $\mathbf{A}_{[K]} \in \mathcal{L}_\kappa^r$ , and thus  $\left\| (\mathbf{A}_{[K]})_{n_{\downarrow}} - \mathbf{A}_{[K]} \right\|_r \rightarrow 0$  for all  $K$ . Whence, by (§) and the continuity of  $\varphi$ , we get  $\psi_\varphi(\mathbf{A}_{[K]}) = \lim_{n \rightarrow \infty} \varphi\left((\mathbf{A}_{[K]})_{n_{\downarrow}}\right) = \varphi(\mathbf{A}_{[K]})$  for all  $K$ . Since by Corollary 3.2.3, we have  $\left\| \mathbf{A}_{[K]} - \mathbf{A} \right\|_r \rightarrow 0$ , it follows from the continuity of  $\psi_\varphi$  and  $\varphi$  that  $\psi_\varphi(\mathbf{A}) = \psi_\varphi\left(\lim_{K \rightarrow \infty} \mathbf{A}_{[K]}\right) = \lim_{K \rightarrow \infty} \psi_\varphi(\mathbf{A}_{[K]}) = \lim_{K \rightarrow \infty} \varphi(\mathbf{A}_{[K]}) = \varphi\left(\lim_{K \rightarrow \infty} \mathbf{A}_{[K]}\right) = \varphi(\mathbf{A})$ . Hence  $\psi_\varphi = \varphi$  on  $\mathcal{L}_\kappa^r$ , which implies that  $\rho_\varphi \in (\mathcal{L}_\kappa^r)^\perp$ . Put  $\mathcal{P} = \{\psi_\varphi : \varphi \in (\mathcal{L}^r)^*\}$ . We claim that  $(\mathcal{L}^r)^* = \mathcal{P} \oplus (\mathcal{L}_\kappa^r)^\perp$  and  $\mathcal{P}$  is isometrically isomorphic to  $(\mathcal{L}_\kappa^r)^*$ . From the definition of  $\mathcal{P}$ , we have already had that  $(\mathcal{L}^r)^* = \mathcal{P} + (\mathcal{L}_\kappa^r)^\perp$ . The decomposition  $(\mathcal{L}^r)^* = \mathcal{P} \oplus (\mathcal{L}_\kappa^r)^\perp$  will be obtained once it can be shown that  $\mathcal{P} \cap (\mathcal{L}_\kappa^r)^\perp = \{0\}$ . To see this, let  $\psi_\varphi \in \mathcal{P} \cap (\mathcal{L}_\kappa^r)^\perp$  for some  $\varphi \in (\mathcal{L}^r)^*$ . Then for every  $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathcal{L}^r$ , we have by the absolute convergence of the series  $\sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$  and the fact that the sequence  $\mathbf{A}_{n_{\downarrow}} \in \mathcal{L}_\kappa^r$  for all  $n$  that  $\psi_\varphi(\mathbf{A}) =$

$\lim_{n \rightarrow \infty} \psi_\varphi(\mathbf{A}_{n_{\downarrow}}) = 0$ . Therefore,  $\psi_\varphi = 0$ , which yields  $\mathcal{P} \cap (\mathcal{L}_\kappa^r)^\perp = \{0\}$ . Accordingly,

we have  $(\mathcal{L}^r)^* = \mathcal{P} \oplus (\mathcal{L}_\kappa^r)^\perp$  as asserted. The rest is to prove that  $\mathcal{P}$  is isometrically isomorphic to  $(\mathcal{L}_\kappa^r)^*$ . To get this, we need to show first that  $\|\psi_\varphi|_{\mathcal{L}_\kappa^r}\| = \|\psi_\varphi\|$ . It is obvious that  $\|\psi_\varphi|_{\mathcal{L}_\kappa^r}\| \leq \|\psi_\varphi\|$ . To have that  $\|\psi_\varphi|_{\mathcal{L}_\kappa^r}\| \geq \|\psi_\varphi\|$ , let  $\epsilon > 0$  be given. Then there is a sequence  $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathcal{L}^r$  such that  $\|\mathbf{A}\|_r \leq 1$  and  $\|\psi_\varphi\| < |\psi_\varphi(\mathbf{A})| + \epsilon$ . Thus, by the absolute convergence of the series  $\sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$ ,

there is a positive integer  $n$  such that  $\|\psi_\varphi\| < |\psi_\varphi(\mathbf{A}_{n_{\downarrow}})| + \epsilon < \|\psi_\varphi|_{\mathcal{L}_\kappa^r}\| + \epsilon$  for all  $\epsilon > 0$ . This implies that  $\|\psi_\varphi\| \leq \|\psi_\varphi|_{\mathcal{L}_\kappa^r}\|$ , and hence we obtain  $\|\psi_\varphi|_{\mathcal{L}_\kappa^r}\| = \|\psi_\varphi\|$  as desired. From this result, the map  $\psi_\varphi \mapsto \psi_\varphi|_{\mathcal{L}_\kappa^r}$  is now an isometric isomorphism from  $\mathcal{P}$  into  $(\mathcal{L}_\kappa^r)^*$ . To see that it is onto, let  $\varphi_0 \in (\mathcal{L}_\kappa^r)^*$ . We then have by the Hahn Banach extension theorem that  $\varphi_0$  can extend uniquely to a bounded linear functional  $\varphi$  on  $\mathcal{L}^r$  with  $\|\varphi\| = \|\varphi_0\|$ . Since  $\psi_\varphi$  agrees with  $\varphi$  on  $\mathcal{L}_\kappa^r$ , it follows  $\psi_\varphi|_{\mathcal{L}_\kappa^r} = \varphi_0$ . Consequently, the map  $\psi_\varphi \mapsto \psi_\varphi|_{\mathcal{L}_\kappa^r}$  is an isometric isomorphism from  $\mathcal{P}$  onto  $(\mathcal{L}_\kappa^r)^*$ .

(3). Let  $\varphi = \psi_\varphi + \rho_\varphi \in (\mathcal{L}^r)^*$ . It is apparent that  $\|\varphi\| \leq \|\psi_\varphi\| + \|\rho_\varphi\|$ . We want to show that the reverse inequality holds. To prove this, let  $\epsilon > 0$  be given. Then there is a sequence  $\mathbf{A} = \{A_k\}_{k=1}^\infty \in \mathcal{L}^r$  with  $\|\mathbf{A}\|_r \leq 1$  such that  $|\psi_\varphi(\mathbf{A})| > \|\psi_\varphi\| - \frac{\epsilon}{3}$ . From this, we have by the absolute convergence of the series  $\sum_{k=1}^\infty \sum A_k \bullet B_k^{(\varphi)}$

that there exists a positive integer  $N$  such that  $|\psi_\varphi(\mathbf{A}_{N_{\downarrow}})| > \|\psi_\varphi\| - \frac{\epsilon}{3}$ . Let  $\mathbf{C} = (\text{sgn} \psi_\varphi(\mathbf{A}_{N_{\downarrow}})) \mathbf{A}_{N_{\downarrow}}$ . Then  $\|\mathbf{C}\|_r = \|\mathbf{A}_{N_{\downarrow}}\|_r \leq \|\mathbf{A}\|_r \leq 1$  and  $\psi_\varphi(\mathbf{C}) = |\psi_\varphi(\mathbf{A}_{N_{\downarrow}})| > \|\psi_\varphi\| - \frac{\epsilon}{3}$ . Next, let  $\mathbf{D} = \{D_k\}_{k=1}^\infty \in \mathcal{L}^r$  be such that  $\|\mathbf{D}\|_r \leq 1$ ,  $\rho_\varphi(\mathbf{D}) > 0$  and

$\rho_\varphi(\mathbf{D}) > \|\rho_\varphi\| - \frac{\epsilon}{3}$ . Then by the absolute convergence of the series  $\sum_{k=1}^\infty \sum D_k \bullet B_k^{(\varphi)}$ , we

have  $\psi_\varphi(\mathbf{D}_{n_r}) \rightarrow 0$ . Whence there is a positive integer  $J > N$  such that  $|\psi_\varphi(\mathbf{D}_{J_r})| < \frac{\epsilon}{3}$ . Since  $\mathbf{D} - \mathbf{D}_{J_r} \in \mathcal{L}_\kappa^r$ , it follows that  $\rho_\varphi(\mathbf{D}_{J_r}) = \rho_\varphi(\mathbf{D})$ . Let  $\mathbf{E} = \mathbf{C} + \mathbf{D}_{J_r}$ . Then

$\mathbf{E} \in \mathcal{L}^r$  and by Theorem 1.1(4), we have  $\|\mathbf{E}\|_r = \max\{\|\mathbf{C}\|_r, \|\mathbf{D}_{J_r}\|_r\} \leq 1$ . Thus

$$\begin{aligned}
\|\varphi\| &\geq |\varphi(\mathbf{E})| = |\psi_\varphi(\mathbf{E}) + \rho_\varphi(\mathbf{E})| \\
&= |\psi_\varphi(\mathbf{C}) + \psi_\varphi(\mathbf{D}_{J_r}) + \rho_\varphi(\mathbf{C}) + \rho_\varphi(\mathbf{D}_{J_r})| \\
&= |\psi_\varphi(\mathbf{C}) + \psi_\varphi(\mathbf{D}_{J_r}) + \rho_\varphi(\mathbf{D})| \\
&\geq \psi_\varphi(\mathbf{C}) + \rho_\varphi(\mathbf{D}) - |\psi_\varphi(\mathbf{D}_{J_r})| \\
&> \|\psi_\varphi\| - \frac{\epsilon}{3} + \|\rho_\varphi\| - \frac{\epsilon}{3} - \frac{\epsilon}{3} \\
&= \|\psi_\varphi\| + \|\rho_\varphi\| - \epsilon.
\end{aligned}$$

Since  $\epsilon > 0$  was given arbitrarily, it follows that  $\|\psi_\varphi\| + \|\rho_\varphi\| \leq \|\varphi\|$ . Hence the equation  $\|\varphi\| = \|\psi_\varphi\| + \|\rho_\varphi\|$  is obtained.  $\square$

**Remark 3.2.6.** Since  $(\mathcal{L}_\kappa^r)^*$  is isometrically isomorphic to  $\mathcal{P}$ , we may treat  $(\mathcal{L}_\kappa^r)^*$  as a subspace of  $(\mathcal{L}^r)^*$ . Thus Theorem 3.2.5 can be symbolized analogously to Dixmier's theorem as follows:

$$(\mathcal{L}^r)^* = (\mathcal{L}_\kappa^r)^* \oplus_1 (\mathcal{L}_\kappa^r)_s.$$



# Chapter 4

## Conclusion

Let  $\mathcal{M}_\infty$  be the set of all infinite complex matrices. For each  $1 \leq r < \infty$ , we define a class of sequences of infinite complex matrices  $\mathcal{L}^r$  as follows:

$$\mathcal{L}^r = \left\{ \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^\infty \subseteq \mathcal{M}_\infty : \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right] \in \mathcal{B}(l^2) \right\},$$

and for any sequence  $\left\{ [a_{ji}^{(k)}] \right\}_{k=1}^\infty$  in  $\mathcal{M}_\infty$ , we define

$$\left\| \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^\infty \right\|_r = \begin{cases} \left\| \left[ \sum_{k=1}^\infty |a_{ji}^{(k)}|^r \right] \right\|^{1/r} & \text{if } \left\{ [a_{ji}^{(k)}] \right\}_{k=1}^\infty \in \mathcal{L}^r, \\ \infty & \text{otherwise.} \end{cases}$$

In this thesis, we study some elementary properties and provide some results on the duality of  $\mathcal{L}^r$ . The main goal is to decompose the dual space  $(\mathcal{L}^r)^*$  of  $\mathcal{L}^r$  as an  $l^1$  direct-sum of its two closed subspaces by a way analogous to a beautiful theorem of Dixmier on decomposing the dual  $\mathcal{B}(l^2)^*$  of  $\mathcal{B}(l^2)$ . The following are the results.

We first obtain some characterizations of the sets  $\mathcal{L}^r$ .

**Theorem 4.1.** *Let  $\{A_k\}_{k=1}^\infty$  be a sequence in  $\mathcal{M}_\infty$  and  $1 \leq r < \infty$ . Then the following are equivalent:*

(1)  $\{A_k\}_{k=1}^\infty$  belongs to  $\mathcal{L}^r$ ;

(2)  $A_k \in \mathcal{S}_{2,2}^r(\mathbb{C})$  for all  $k$  and the sequence  $\left\{ \sum_{k=1}^n A_k^{[r]} \right\}_{n=1}^\infty$  is bounded in  $\mathcal{B}(l^2)$ ;

(3) the sequence  $\left\{ \left\| \sum_{k=1}^n A_k^{[r]} \right\| \right\}_{n=1}^\infty$  is bounded.

To obtain that  $\|\cdot\|_r$  is precisely a norm on  $\mathcal{L}^r$ , the following Hölder-type inequality is constructed.

**Theorem 4.2.** (Hölder-type inequality for sequences of matrices) *For any sequences  $\{A_k\}_{k=1}^\infty$  and  $\{B_k\}_{k=1}^\infty$  in  $\mathcal{M}_\infty$ ,*

$$\|\{A_k \bullet B_k\}_{k=1}^\infty\|_1 \leq \|\{A_k\}_{k=1}^\infty\|_r \|\{B_k\}_{k=1}^\infty\|_{r^*},$$

where  $1 < r < \infty$  with  $\frac{1}{r} + \frac{1}{r^*} = 1$ , under the conventions that  $\infty \cdot 0 = 0 \cdot \infty = 0$ ,  $\infty \cdot \alpha = \alpha \cdot \infty = \infty$  for all positive real number  $\alpha$  and  $\infty \cdot \infty = \infty$ .

From the Hölder-type inequality, the corresponding Minkowski's inequality is obtained.

**Theorem 4.3.** (Minkowski-type inequality for sequences of matrices) *For any sequences  $\{A_k\}_{k=1}^\infty$  and  $\{B_k\}_{k=1}^\infty$  in  $\mathcal{M}_\infty$  and  $1 \leq r < \infty$ ,*

$$\|\{A_k + B_k\}_{k=1}^\infty\|_r \leq \|\{A_k\}_{k=1}^\infty\|_r + \|\{B_k\}_{k=1}^\infty\|_r,$$

under the conventions that  $\infty + \alpha = \alpha + \infty = \infty$  for all non-negative real number  $\alpha$  and  $\infty + \infty = \infty$ .

From the Minkowski's inequality, we have that the set  $\mathcal{L}^r$  equipped with the norm  $\|\cdot\|_r$  is a normed space. A Rieze-fischer-type theorem for completeness of the sequence spaces  $\mathcal{L}^r$  is obtained below.

**Theorem 4.4.** *For each  $1 \leq r < \infty$ , the set  $\mathcal{L}^r$  equipped with the norm  $\|\cdot\|_r$  is a Banach space.*

For the classical sequence spaces  $l^p$  ( $1 \leq p < \infty$ ), there is a result closely related to their duality stating that for every  $1 \leq p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , a sequence  $\{x_k\}_{k=1}^\infty$  belongs to  $l^q$  if and only if  $\{x_k y_k\}_{k=1}^\infty \in l^1$  for all  $\{y_k\} \in l^p$ . We obtain a similar duality-type result for the sequence spaces  $\mathcal{L}^r$  as follows.

**Theorem 4.5.** *Let  $1 < r < \infty$  with  $\frac{1}{r} + \frac{1}{r^*} = 1$ .*

- (1) *A sequence  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^{r^*}$  if and only if  $\{A_k \bullet B_k\}_{k=1}^\infty \in \mathcal{L}^1$  for all  $\{B_k\}_{k=1}^\infty \in \mathcal{L}^r$ .*
- (2) *If  $\{A_k\}_{k=1}^\infty \in \mathcal{L}^{r^*}$ , then*

$$\|\{A_k\}_{k=1}^\infty\|_{r^*} = \sup\{\|\{A_k \bullet B_k\}_{k=1}^\infty\|_1 : \{B_k\}_{k=1}^\infty \in \mathcal{L}^r, \|\{B_k\}_{k=1}^\infty\|_r \leq 1\}.$$

Next, we define the class

$$\mathcal{L}_\kappa^r = \left\{ \left\{ \left[ a_{ji}^{(k)} \right] \right\}_{k=1}^\infty \subseteq \mathcal{M}_\infty : \left[ \sum_{k=1}^\infty \left| a_{ji}^{(k)} \right|^r \right] \in \mathcal{K}(l^2) \right\},$$

We obtain a characterization of  $\mathcal{L}_\kappa^r$  as follows.

**Theorem 4.6.** Let  $\{A_k\}_{k=1}^\infty$  be a sequence in  $\mathcal{M}_\infty$  and  $1 \leq r < \infty$ . Then  $\{A_k\}_{k=1}^\infty \in \mathcal{L}_\kappa^r$  if and only if  $A_k^{[r]}$  is compact for all  $k$  and the sequence  $\left\{ \sum_{k=1}^n A_k^{[r]} \right\}_{k=1}^\infty$  converges in  $\mathcal{B}(l^2)$ .

**Theorem 4.7.** For each  $1 \leq r < \infty$ , the set  $\mathcal{L}_\kappa^r$  is a Banach subspace of  $\mathcal{L}^r$ .

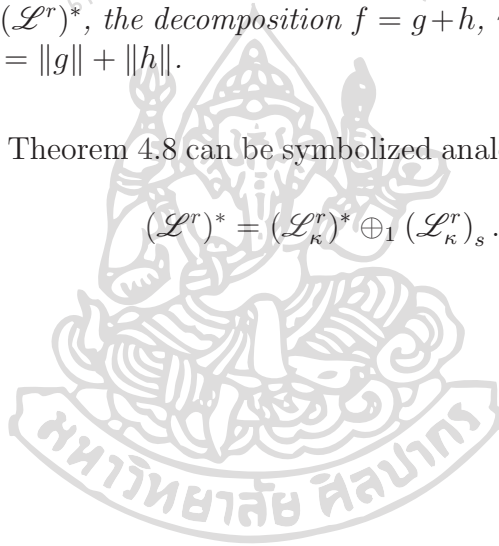
We finally obtain a decomposition theorem for the dual  $(\mathcal{L}^r)^*$  of  $\mathcal{L}^r$  as follows.

**Theorem 4.8.** Let  $1 \leq r < \infty$ . Then the following hold.

- (1) The annihilator  $(\mathcal{L}_\kappa^r)^\perp$  of  $\mathcal{L}_\kappa^r$  is a non-trivial closed subspace of the dual  $(\mathcal{L}^r)^*$  of  $\mathcal{L}^r$ .
- (2) There is a subspace  $\mathcal{P}$  of  $(\mathcal{L}^r)^*$  such that  $\mathcal{P}$  is isometrically isomorphic to  $(\mathcal{L}_\kappa^r)^*$  and  $(\mathcal{L}^r)^* = \mathcal{P} \oplus (\mathcal{L}_\kappa^r)^\perp$ .
- (3) For any  $f \in (\mathcal{L}^r)^*$ , the decomposition  $f = g + h$ , where  $g \in \mathcal{P}$  and  $h \in (\mathcal{L}_\kappa^r)^\perp$ , satisfies  $\|f\| = \|g\| + \|h\|$ .

Notice that Theorem 4.8 can be symbolized analogously to Dixmier's theorem as follows:

$$(\mathcal{L}^r)^* = (\mathcal{L}_\kappa^r)^* \oplus_1 (\mathcal{L}_\kappa^r)_s.$$



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