SEQUENCE SPACES OF MATRICES OVER C^*- ALGEBRAS

By

Aveya Charoenpol

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

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ปริภูมิลำดับของเมทริกซ์เหนือพีชคณิตC^*

โดย

นางสาวอาวีญา เจริญผล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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The Graduate School, Silpakorn University has approved and accredited the Thesis title of “Sequence spaces of metrices over C^*-algebra” submitted by Miss Aveya Charoenpol as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Let $\mathcal{A}$ be a commutative unital C^*- algebra with the state space $\mathcal{s}(\mathcal{A})$. Let $\mathcal{S}^2(\mathcal{A})$ be the set of all matrices $A = [a_{ji}]$ over $\mathcal{A}$ such that the matrix $\varphi(A^{[2]}) = [\varphi(a_{ji}{}^* a_{ji})] \in \mathcal{B}(l_2)$ for all $\varphi \in \mathcal{s}(\mathcal{A})$. In this paper, we define some new sequence spaces of infinite matrices in $\mathcal{S}^2(\mathcal{A})$ by a way analogous to the sequence spaces of matrices of complex numbers provided in the paper titled "Sequence spaces of operators on l_2 " by J. Rakbud and S.-C. Ong contributed in 2010, and discuss some of their basic properties.

Department of Mathematics Graduate School, Silpakorn University Academic Year 2010

Student's signature ....

Thesis Advisor's signature ....

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คำสำคัญ : พีชคณิตบานาค / พีชคณิต C^* / ปริภูมิบานาค / สเตต

อาวีญา เจริญผล : ปริภูมิลำดับของเมทริกซ์เหนือพีชคณิต C^*. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : ดร. จิตติ รักบุตร. 49 หน้า.

ให้ $\mathcal{A}$ เป็นพีชคณิต C^* สลับที่ซึ่งมีเอกลักษณ์ และ $\mathcal{S}(\mathcal{A})$ เป็นปริภูมิสเตต และให้ $\mathcal{S}^2(\mathcal{A})$ เป็นเซตของเมทริกซ์ $A = [a_{ji}]$ เหนือ $\mathcal{A}$ ซึ่ง $\varphi\left(A^{[2]}\right) = [\varphi({a_{ji}}^* a_{ji})] \in \mathcal{B}(l_2)$ สำหรับทุกๆ $\varphi \in \mathcal{S}(\mathcal{A})$ ในวิทยานิพนธ์นี้ เราได้นิยาม และศึกษาสมบัติพื้นฐานบางประการของปริภูมิลำดับของเมทริกซ์ใน $\mathcal{S}^2(\mathcal{A})$ ซึ่งถูกนิยามในทำนองเดียวกับ ปริภูมิลำดับของจำนวนเชิงซ้อน ที่นิยามในผลงานวิจัยเรื่อง " Sequence spaces of operators on l_2" โดย J. Rakbud และ S.-C. Ong

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ลายมือชื่อนักศึกษา..

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Chapter 1

Introduction and Literature Review

The classical Banach spaces l_p $(1 \leq p \leq \infty)$ have played an important role for a century in many branches of mathematics such as functional analysis, matrix analysis, operator theory, and operator algebras. Currently, "theory of sequence spaces"is a branch of mathematics which has been widely studied. There have been a lot of research works on this area dealing with generalizing the classical sequence spaces l_p. The most natural generalization is the following sequence space defined over any Banach space X:

$$l_p(X) = \left\{ \{x_k\}_{k=1}^{\infty} \subseteq X : \sum_{k=1}^{\infty} \|x_k\|^p < \infty \right\}.$$

Some geometric properties, sequential convergence, duality and reflexivity of these spaces were studied by I. E. Leonard in [9] since 1976. Another sequence space which also generalizes the spaces l_p naturally, but with a condition weaker than the one above, is the following:

$$l_p[X] = \left\{ \{x_k\}_{k=1}^{\infty} \subseteq X : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \text{ for all } f \in X^* \right\}.$$

The study on these sequence spaces has been contributed by many people (see [1], [5], [16] for references).

If $\mathscr{A}$ is a commutative C^*-algebra with identity, then by Gelfand-Naimark's theorem, there exists a compact Hausdorff space X such that $\mathscr{A}$ is isometrically *-isomorphic to $C(X)$, the C^*- algebra of complex-valued functions on X. In 2004, L. Livshits, S.-C. Ong, and S.-W. Wang defined in [11] sequence spaces over a commutative C^*-algebra $\mathscr{A} \cong^* C(X)$ by a way analogous to the Hilbert space l_2 as follows:

$$l_2^b(C(X)) = \left\{ \{f_k\}_{k=1}^{\infty} \subseteq C(X) : \left\{ \sum_{k=1}^{n} |f_k|^2 \right\}_{n=1}^{\infty} \text{ is bounded in } C(X) \right\};$$

$$l_2^c(C(X)) = \left\{ \{f_k\}_{k=1}^{\infty} \subseteq C(X) : \left\{ \sum_{k=1}^{n} |f_k|^2 \right\}_{n=1}^{\infty} \text{ converges in } C(X) \right\}.$$

In [14], J. Rakbud and P. Chaisuriya extended those sequence spaces to any $1 \leq p < \infty$ and studied the duality, preduality, and reflexivity of them. In 2009, S. Leelahanon and P. Chaisuriya extended in [8] the results of Rakbud and Chaisuriya mentioned above. They defined two sequence spaces over any C^*-algebras $\mathscr{A}$ with identity, in particular the non-commutative C^*-algebra of bounded linear operators on l_2, as follows:

$$\mathscr{L}_b(\mathscr{A}) = \left\{ \{a_k\}_{k=1}^{\infty} \subseteq \mathscr{A} : \left\{ \sum_{k=1}^{n} |a_k|^2 \right\}_{n=1}^{\infty} \text{ is bounded in } \mathscr{A} \right\};$$

$$\mathscr{L}_c(\mathscr{A}) = \left\{ \{a_k\}_{k=1}^{\infty} \subseteq \mathscr{A} : \left\{ \sum_{k=1}^{n} |a_k|^2 \right\}_{n=1}^{\infty} \text{ converges in } \mathscr{A} \right\},$$

where for any $a \in \mathscr{A}$, $|a|$ is the positive square root a^*a.

Let $1 \leq p, q < \infty$. For any infinite scalar matrix $A = [a_{jk}]$, we say that A defines a linear operator from l_p into l_q if for each $x = \{x_k\}_{k=1}^{\infty}$ in l_p the series $\sum_{k=1}^{\infty} a_{jk}x_k$ converges for all j, and the sequence $Ax := \left\{ \sum_{k=1}^{\infty} a_{jk}x_k \right\}_{j=1}^{\infty}$ belongs to l_q. If A defines a linear operator from l_p into l_q, we call the operator $x \mapsto Ax$ the *linear operator defined by A*. By the uniform boundedness principle, it can be shown that the linear operator defined by A is bounded. Let $\mathcal{B}(l_p, l_q)$ be the set of all infinite matrices defining linear operators from l_p into l_q. For the case where $p = q$, we denote $\mathcal{B}(l_p, l_p)$ by just $\mathcal{B}(l_p)$. For any matrix A, we define $\|A\|_{p,q}$ to be the norm of the linear operator defined by A if $A \in \mathcal{B}(l_p, l_q)$, and to be ∞ otherwise. It is well known that $\mathcal{B}(l_p, l_q)$ equipped with the norm $\|\cdot\|_{p,q}$ is a Banach space. In deed, It is exactly the set of matrix representations of all bounded linear operators from l_p into l_q with respect to the standard Schauder bases on l_p and l_q. For any matrix A, we call A *bounded* if $A \in \mathcal{B}(l_p, l_q)$ and *unbounded* otherwise. It is well known that a matrix A belongs to $\mathcal{B}(l_p, l_q)$ if and only if $\sup_n \|A_{n\lrcorner}\|_{p,q} < \infty$, where $A_{n\lrcorner}$ is the matrix whose entries in the upper left $n \times n$-block are exactly those of A and all other entries are zeros, and also known that for any matrix A, $\|A_{n\lrcorner}\|_{p,q} \nearrow \|A\|_{p,q}$. A matrix A is said to be *compact* if the linear operator defined by A is compact. It is well known that a matrix A is compact as an operator on l_2 if and only if $\|A_{n\lrcorner} - A\|_{2,2} \to 0$.

The *Schur product* (also known as the *Hardamard product* or *entry-wise product*) of two scalar matrices $A = [a_{jk}]$ and $B = [b_{jk}]$ of the same size is the matrix $A \bullet B := [a_{jk}b_{jk}]$. In [15], Schur proved that $\mathcal{B}(l_2)$ is a commutative Banach algebra under the operator norm and the Schur product multiplication. In [2], G. Bennett extended the result from the study of Schur mentioned above. He proved that for each $1 \leq p, q < \infty$, $\mathcal{B}(l_p, l_q)$ is also a Banach algebra under this simple multiplication. In [3] P. Chaisuriya and S.-C. Ong studied the class of matrices over any Banach algebra with identity. In that paper, for a fixed Banach algebra $\mathscr{B}$ with identity and $1 \leq p, q, r < \infty$, the authors defined the class $\mathcal{S}^r_{p,q}(\mathscr{B})$ of matrices $A = [a_{jk}]$ over $\mathscr{B}$ such that *the absolute Schur rth-power* $A^{[r]} := [\|a_{jk}\|^r]$

defines a linear operator from l_p into l_q. And they proved that it is a Banach algebra under the *the absolute Schur r-norm* defined by $|||A|||_{p,q,r} = \left\|A^{[r]}\right\|_{p,q}^{1/r}$ and the Schur product which is generalized to this setting via the product in the Banach algebra $\mathscr{B}$. The authors also gave a nice relationship between the class $\mathcal{B}(l_p, l_q)$ of all bounded operators and the algebra $\mathcal{S}^2_{p,q}(\mathbb{C})$. They found that $\mathcal{S}^2_{p,q}(\mathbb{C})$ contains $\mathcal{B}(l_p, l_q)$ as a non-closed ideal.

From the virtue of the absolute Schur algebra $\mathcal{S}^2_{2,2}(\mathbb{C})$ that it contains the set of all bounded linear operators on l_2 as an ideal, J. Rakbud and S.-C. Ong defined in [13] some sequence spaces of matrices from $\mathcal{S}^2_{2,2}(\mathbb{C})$ (see [13]). The following are those sequence spaces:

$$\mathscr{O}_b = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2_{2,2}(\mathbb{C}) : \text{ the sequence } \left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty} \text{ is bounded in } \mathcal{B}(l_2) \right\};$$

$$\mathscr{O}_c = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2_{2,2}(\mathbb{C}) : \text{ the sequence } \left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{B}(l_2) \right\};$$

and

$$\mathscr{O}_\kappa := \left\{ \left\{ A_k = \left[a_{ji}^{(k)}\right] \right\}_{k=1}^{\infty} \subset \mathcal{S}^2_{2,2}(\mathbb{C}) : \text{ the matrix } \left[\sum_{k=1}^{\infty} \left| a_{ji}^{(k)} \right|^2 \right] \text{ is compact} \right\}.$$

The authors proved that $\mathscr{O}_\kappa \subsetneq \mathscr{O}_c \subsetneq \mathscr{O}_b$. They defined a norm on these three spaces by

$$|||\{A_k\}_{k=1}^{\infty}||| = \left(\sup_n \left\| \sum_{k=1}^{n} A_k^{[2]} \right\| \right)^{1/2}.$$

and proved that the three sequence spaces equipped with this norm are Banach spaces. It was noticed that due to the non-closedness of $\mathcal{B}(l_2)$ in $\mathcal{S}^2_{2,2}(\mathbb{C})$, the restrictions of these sequence spaces to $\mathcal{B}(l_2)$ are not complete. In addition, the authors also studied sequential convergence and duality in the three sequence spaces.

In 2007, the absolute Schur algebras $\mathcal{S}^r_{p,q}(\mathbb{C})$ were generalized in [4] by P. Chaisuriya, S.-C. Ong and S.-W. Wang to the setting of matrices over a fixed C^*-algebra with identity. The authors defined for each $1 \leq r, p, q < \infty$ the set $\mathcal{S}^r_{p,q}(\mathscr{A})$ of matrices over a fixed C^*-algebra $\mathscr{A}$ with identity by

$$\mathcal{S}^r_{p,q}(\mathscr{A}) = \{[a_{ji}] : [\varphi(|a_{ji}|^r)] \in \mathcal{B}(l_p, l_q) \text{ for all state } \varphi \text{ of } \mathscr{A}\},$$

where for any $a \in \mathscr{A}$, $|a|^r$ is the image of the function f defined on the spectrum of $|a|$ by $f(t) = t^r$, under the function calculus of $|a|$. They also studied the completeness under a suitable norm and preduality and duality of the sets $\mathcal{S}^r_{p,q}(\mathscr{A})$

In this research we extend some results of J. Rakbud and S.-C. Ong mentioned above to the setting of matrices in the absolute Schur algebra $\mathcal{S}^2_{2,2}(\mathscr{A})$ over a fixed commutative C^*-algebra $\mathscr{A}$ with identity. The following are our objectives.

(1) We first define an appropriate class of matrices over a fixed C^*-algebra $\mathscr{A}$ with identity for playing the role as the Banach space $\mathcal{B}(l_2)$ in the study of Rakbud and Ong mentioned above.

(2) We generalize the three sequence spaces $\mathscr{O}_b$, $\mathscr{O}_c$, and $\mathscr{O}_k$ to the setting of matrices in the the absolute Schur algebra $\mathcal{S}^2_{2,2}(\mathscr{A})$.

(3) We study some properties of those sequence spaces.

Chapter 2

Theoretical Background

In this chapter, we provide some theoretical background related to the main results, which are presented in the next chapter. All theorems here are stated without proofs.

2.1 Topological spaces

Definition 2.1.1. [10] A *topology* on a set X is a collection τ of subsets of X having the following properties:

(i) ϕ and X are in τ.

(ii) The union of the elements of any subcollection of τ is in τ.

(iii) The intersection of the elements of any finite subcollection of τ is in τ.

A pair (X, τ) of a non-empty set X and a topology τ on X is called a *topological space*. In the unobscure situation of what topology on X which is being considered, we may write just X is a topological space. If (X, τ) is a topological space, we say that a subset U of X is an *open set* of (X, τ) if U belongs to the collection τ. A subset A of a topological space (X, τ) is *closed* if $X - A$ is open. A subset U of a topology space (X, τ) is called a *neighborhood of a point* $x \in X$ if there is $V \in \tau$ such that $x \in V \subseteq U$.

Definition 2.1.2. [10] A relation $\succeq$ on set D is called a *partial order* relation if for every α, β, γ in D, the following conditions hold:

(i) $\alpha \succeq \alpha$.

(ii) If $\alpha \succeq \beta$ and $\beta \succeq \alpha$, then $\alpha = \beta$.

(iii) If $\alpha \succeq \beta$ and $\beta \succeq \gamma$, then $\alpha \succeq \gamma$.

Definition 2.1.3. [10] A *directed set* D is a set with a partial order $\succeq$ such that for each pair α, β of elements of D, there exists an element γ of D having the property that $\gamma \succeq \alpha$ and $\gamma \succeq \beta$.

Definition 2.1.4. [10] Let X be a topological space. A *net* in X is a function f from a directed set D into X. If $\alpha \in D$, we usually denote $f(\alpha)$ by x_α. We denote the net f itself by the symbol $\{x_\alpha\}_{\alpha \in D}$, or merely by $\{x_\alpha\}$ if the index set is understood.

Definition 2.1.5. [10] The net $\{x_\alpha\}$ is said to *converge* to the point x of X (written $x_\alpha \to x$) if for each neighborhood U of x, there exists $\alpha \in J$ such that $x_\beta \in U$ for all $\beta \in D$ with $\beta \succeq \alpha$.

Definition 2.1.6. [10] Let X and Y be topological spaces and let f is a function from X into Y. We say that f is *continuous* if $f^{-1}(U)$ is open in X for all open set U in Y.

Theorem 2.1.7. [10] *Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*

(1) *f is continuous.*

(2) *For each $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.*

(3) *For every convergent net $\{x_\alpha\}$ in X converging to x, the net $\{f(x_\alpha)\}$ converges to $f(x)$.*

Definition 2.1.8. [7] A topological space X is called a *Hausdorff space* if for every two different points x and y in X, there are neighborhoods U and V of x and y respectively such that $U \cap V = \emptyset, x \notin V$ and $y \notin U$.

Definition 2.1.9. [7] A topological space X is called a *compact space* if every open cover, a family of $\{U_i : i \in I\}$ of open sets of X with $X = \bigcup_{i \in I} U_i$, has a finite subcover.

2.2 Banach spaces

Throughout this thesis, we let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ to denote the set of all complex numbers, real numbers, and natural numbers respectively.

Definition 2.2.1. [12] Let X be a vector space over a scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A *norm* on X is a real-valued function $\|\cdot\|$ on X satisfying the following properties:

(i) $\|x\| \geq 0$;

(ii) $\|x\| = 0 \iff x = 0$;

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$;

(iv) $\|x + y\| \leq \|x\| + \|y\|$ (*Triangle inequality*), where x and y are arbitrary vectors in X and α is any scalar in $\mathbb{K}$. A *normed space* is a pair $(X, \|\cdot\|)$ of a non-empty set X and a norm $\|\cdot\|$ on X. It may be sometimes written just X as a normed space by omitting the norm on X.

Definition 2.2.2. [12] A sequence $\{x_n\}_{n=1}^{\infty}$ in a norm space X is said to *converge* or to be *convergent* if there is a point x in X which for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\|x - x_n\| < \epsilon \quad \text{for all } \; n \geq N.$$

The point x is called the *limit* of $\{x_n\}_{n=1}^{\infty}$ and write $\lim\limits_{n\to\infty} x_n = x$, or simply, $x_n \to x$.

Definition 2.2.3. [12] A sequence $\{x_n\}_{n=1}^{\infty}$ in a norm space X is said to be a *bounded* in X if there is a positive real number c such that $\|x_n\| \leq c$ for all $n \in \mathbb{N}$.

Definition 2.2.4. [12] A sequence $\{x_n\}_{n=1}^{\infty}$ in a norm space X is said to be a *Cauchy sequence* in X if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\|x_m - x_n\| < \epsilon$$

for all $m, n \geq N$. A norm space X is said to be a *Banach space* if it is *complete* under the metric d defined by $d(x, y) = \|x - y\|$.

Definition 2.2.5. [12] Let X and Y be vector spaces having the same scalar field. A function $T : X \to Y$ is said to be a *linear operator* or *linear function* or *linear transformation* if

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$$

for every $x_1, x_2 \in X$ and any scalar α and β.

Definition 2.2.6. [12] Let X and Y be normed spaces having the same scalar field. A linear operator $T : X \to Y$ is said to be *bounded* if $T(B)$ is bounded for all bounded subset B of X.

Theorem 2.2.7. *Let $T : X \to Y$ be a linear operator from a normed space X into a normed space Y. Then the following are equivalent.*

(1) *T is bounded.*

(2) *T is continuous.*

(3) *There is a constant $M > 0$ such that $\|Tx\| \leq M \|x\|$ for all $x \in X$.*

Let $\mathcal{B}(X, Y)$ be the set of all bounded linear operators from a normed space X into a normed space Y.

Definition 2.2.8. [12] Let X and Y be normed spaces. For each T in $\mathcal{B}(X, Y)$, the *norm* or *operator norm* $\|T\|$ of T is the nonnegative real number $\sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$. The operator norm on $\mathcal{B}(X, Y)$ is the map $T \mapsto \|T\|$.

Corollary 2.2.9. [12] *If T is a bounded linear operator from a normed space X into a normed space Y, then $\|Tx\| \leq \|T\| \|x\|$ for all x in X. Furthermore, the number $\|T\|$ is the smallest nonnegative real number M such that $\|Tx\| \leq M \|x\|$ for all $x \in X$.*

Theorem 2.2.10. *If X is a normed space and Y is a Banach spaces, then the set $\mathcal{B}(X, Y)$ equipped with the operator norm is a Banach space.*

Theorem 2.2.11. [12] (The Uniform Boundedness Principle) *Let $\mathcal{F}$ be a nonempty family of bounded linear operators from a Banach space X into a normed space Y. If* $\sup\{\|Tx\| : T \in \mathcal{F}\}$ *is finite for each x in X, then* $\sup\{\|T\| : T \in \mathcal{F}\}$ *is finite.*

Definition 2.2.12. [12] Let X and Y be normed spaces and let $T : X \to Y$ be a linear operator. Then a norm on the cartesian product $X \times Y$ can be defined as follows:

$$\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

The normed space $(X \times Y, \|\cdot\|)$ is denoted specifically by $X \oplus_{l_2} Y$. *The graph of* T denoted by $G(T)$ is the subset of $X \times Y$ consisting of all order pairs of the form (x, Tx) as x varies over X, that is, $G(T) = \{(x, Tx) : x \in X\}$.

Theorem 2.2.13. [12] *Let X and Y be normed spaces and let $T : X \to Y$ be a linear operator.*

(1) *If X and Y are Banach space, then so is $X \oplus_{l_2} Y$.*

(2) *The graph $G(T)$ is a normed subspace of $X \oplus_{l_2} Y$.*

(3) *The graph $G(T)$ is a closed in $X \oplus_{l_2} Y$ if and only if whenever we have $x_n \to x$ in X and $Tx_n \to y$ in Y, it follows that $Tx = y$.*

Theorem 2.2.14. [12] (The Closed Graph Theorem) *Let X and Y be Banach spaces, and let $T : X \to Y$ be a linear operator. Then T is bounded if and only if its graph is closed in $X \oplus_{l_2} Y$.*

Definition 2.2.15. [12] A *linear functional* f is a linear operator from a normed space X into the scalar filed $\mathbb{K}$ of X, where $\mathbb{K}$ is regarded as a normed space under the usual norm on $\mathbb{K}$.

If X is a normed space, then the set of all bounded linear functionals on X is denoted by X^*. By Theorem 2.2.10, the normed space X^* is immediately a Banach space.

Definition 2.2.16. [12] Let X be a normed space. The *weak topology* on X is the smallest topology on X that makes all linear functionals in X^* continuous.

Definition 2.2.17. [12] Let X be a normed space and $\widehat{X}$ the set of all linear functionals $\widehat{x}$ on X^* defined for each $x \in X$ by $\widehat{x}(\rho) = \rho(x)$ for all $\rho \in X^*$. The *weak* topology* on X^* is the smallest topology on X^* which makes all linear functionals in $\widehat{X}$ continuous.

Theorem 2.2.18. [12] *Let X be a Banach space.*

(1) *A net $\{x_\alpha\}$ converges to a point x in X equipped with the weak topology if and only if $f(x_\alpha) \to f(x)$ for all $f \in X^*$.*

(2) *A net* $\{\rho_\alpha\}$ *converges to a point* ρ *in* X^* *equipped with the weak** *topology if and only if* $\rho_\alpha(x) \to \rho(x)$ *for all* $x \in X$.

Definition 2.2.19. [12] Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is *compact* if $\overline{T(B)}$ is compact for all bounded subset B of X.

It is clear that every compact operator is necessarily bounded.

Definition 2.2.20. [12] A linear operator T from a Banach space X into a Banach space Y is said to be *finite rank* if $T(X)$ is finite dimensional.

Theorem 2.2.21. [12] *Every finite rank operator is compact.*

2.3 l_p spaces

Definition 2.3.1. [12] For $1 \leq p \leq \infty$ and sequence $\{\lambda_k\}_{k=1}^{\infty}$ of complex numbers, the *p-norm* of $\{\lambda_k\}_{k=1}^{\infty}$ is defined by

$$\|\{\lambda_k\}_{k=1}^{\infty}\|_p = \begin{cases} \left(\sum_{k=1}^{\infty} |\lambda_k|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\{|\lambda_k| : k = 1, 2, 3, ...\} & \text{if } p = \infty. \end{cases}$$

For each $1 \leq p < \infty$, let

$$l_p = \left\{ \{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sum_{k=1}^{\infty} |\lambda_k|^p < \infty \right\}$$

and

$$l_\infty = \{\{\lambda_k\}_{k=1}^{\infty} \subseteq \mathbb{C} : \sup\{|\lambda_k| : k = 1, 2, 3, ...\} < \infty\}.$$

Theorem 2.3.2. [12] (Hölder's inequality) *For any* $1 \leq p \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and sequences* $\mathbf{x}$ *and* $\mathbf{y}$ *of complex numbers,* $\|\mathbf{xy}\|_1 \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$.

In particular, Hölder's inequality is called Cauchy-Schwartz's inequality if $p = q = 2$. From Hölder's inequality, the following Minkowski's inequality is obtained.

Theorem 2.3.3. [12] (Minkowski's inequality) *For any* $1 \leq p \leq \infty$ *and sequences* $\mathbf{x}$ *and* $\mathbf{y}$ *of complex numbers,* $\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$.

Theorem 2.3.4. *For any* $1 \leq p \leq \infty$*, the set* l_p *endowed with the p-norm* $\|\cdot\|_p$ *is a Banach space.*

For $1 \leq p < \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we define, for each $\mathbf{x} = \{x_k\}_{k=1}^{\infty} \in l_q$, the function $f_{\mathbf{x}} : l_p \to \mathbb{C}$ by

$$f_{\mathbf{x}}(\{y_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} x_k y_k \text{ for all } \{y_k\}_{k=1}^{\infty} \in l_p.$$

By Hölder's inequality, we have the function $f_{\mathbf{x}}$ is well-defined.

Theorem 2.3.5. [12] *Let* $1 \leq p < \infty$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Then* l_q *is isometrically isomorphic to* l_p^* *by the isomorphism defined by* $\mathbf{x} \mapsto f_{\mathbf{x}}$.

The following result is closely related to the dulaity theorem above.

Theorem 2.3.6. [12] *Let* $1 \leq p < \infty$ *with* $1 = \dfrac{1}{p} + \dfrac{1}{q}$. *Then a sequence* $\{x_k\}_{k=1}^{\infty}$ *of complex numbers belongs to* l_q *if and only if* $\{x_k y_k\}_{k=1}^{\infty}$ *belongs to* l_1 *for all* $\{y_k\}_{k=1}^{\infty}$ *in* l_p.

2.4 C^*- algebras

Definition 2.4.1. [6] An *algebra* $\mathscr{A}$ over a scalar field $\mathbb{K}$ is a vector space $\mathscr{A}$ over the scalar field $\mathbb{K}$ equipped with a multiplication, a function $(x, y) \mapsto xy$ from $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$ having the following properties:

(i) $(xy)z = x(yz)$;

(ii) $x(y + z) = xy + xz$;

(iii) $(x + y)z = xz + yz$;

(iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in \mathscr{A}$ and $\alpha \in \mathbb{K}$. The algebra $\mathscr{A}$ is said to be *commutative* if the multiplication is commutative, that is, if for all $x, y \in \mathscr{A}$, $xy = yx$. An element I of an algebra $\mathscr{A}$ is called an *identity* if $Ix = xI = x$ for all $x \in \mathscr{A}$.

Definition 2.4.2. [6] A *normed algebra* $\mathscr{A}$ is a normed space which is an algebra such that for all $x, y \in \mathscr{A}$, $\|xy\| \leq \|x\| \, \|y\|$. A normed algebra having an identity I and $\|I\| = 1$ is called a *normed algebra with identity.* A *Banach algebra* is a normed algebra which is a Banach space.

Definition 2.4.3. [6] Let $\mathscr{A}$ be a complex Banach algebra. By an *involution* on $\mathscr{A}$, we mean a function $x \mapsto x^*$, from $\mathscr{A}$ into $\mathscr{A}$, such that

(i) $\alpha x + \beta y = \overline{\alpha} x^* + \overline{\beta} y^*$;

(ii) $(xy)^* = y^* x^*$;

(iii) $(x^*)^* = x$;

for all $x, y \in \mathscr{A}$ and $\alpha, \beta \in \mathbb{C}$.

Definition 2.4.4. [6] A C^*- *algebra* is a complex Banach algebra $\mathscr{A}$ equipped with an involution on $\mathscr{A}$ satisfying the following additional property:

$$\|x^* x\| = \|x\|^2$$

for all $x \in \mathscr{A}$.

Definition 2.4.5. [6] Let $\mathscr{A}$ be a C^*- algebra and $x \in \mathscr{A}$.

(i) The element x is said to be *self-adjoint* if $x = x^*$.

(ii) The element x is said to be *normal* if $x^*x = xx^*$.

Definition 2.4.6. [6] Let $\mathscr{A}$ be an algebra with the identity I. An element $x \in \mathscr{A}$ is said to be *invertible* if there is an element $x^{-1} \in \mathscr{A}$ such that $xx^{-1} = x^{-1}x = I$. We call x^{-1} the inverse of x.

Definition 2.4.7. [6] Let $\mathscr{A}$ be an algebra with the identity I and $x \in \mathscr{A}$. The *resolvent set* $p(x)$ of $\mathscr{A}$ is the set of all $\alpha \in \mathbb{K}$ such that $\alpha I - x$ is invertible. We call the complement of $p(x)$ the *spectrum of* x and denote by $sp(x)$.

Definition 2.4.8. [6] Let $\mathscr{A}$ be a C^*- algebra with identity. An element x of $\mathscr{A}$ is *positive* if x is self-adjoint and $sp(x) \subseteq [0, \infty)$. The set of all positive elements of $\mathscr{A}$ is denoted by $\mathscr{A}^+$.

Theorem 2.4.9. [6] *If $\mathscr{A}$ is a C^*- algebra with identity and $x \in \mathscr{A}$, then the following conditions are equivalent:*

(1) $x \in \mathscr{A}^+$;

(2) $x = y^2$ *for some* $y \in \mathscr{A}^+$;

(3) $x = z^*z$ *for some* $z \in \mathscr{A}$.

Definition 2.4.10. [6] Let $\mathscr{A}$ be a C^*- algebra with the identity I. A linear functional ρ on $\mathscr{A}$ is said to be *positive* if $\rho(x) \geq 0$ for all $x \in \mathscr{A}^+$. A positive linear functional ρ on $\mathscr{A}$ is called a *state* of $\mathscr{A}$ if $\rho(I) = 1$. Let $\mathscr{S}(\mathscr{A})$ be the set of all states of $\mathscr{A}$. The *state space* of $\mathscr{A}$ is the set $\mathscr{S}(\mathscr{A})$ equipped with the weak* topology.

Theorem 2.4.11. [6] *The state space of a C^*- algebra with identity is a compact Hausdorff space.*

Proposition 2.4.12. [6] *Let ρ be a positive linear functional on a C^*- algebra $\mathscr{A}$ with identity and $x, y \in \mathscr{A}$. Then*

(1) $|\rho(x^*y)|^2 \leq \rho(x^*x)\rho(y^*y)$.

(2) $\rho\left((x+y)^*(x+y)\right)^{1/2} \leq (\rho(x^*x))^{1/2} + (\rho(y^*y))^{1/2}$.

(3) $\rho\left((xy)^*(xy)\right) \leq \|x\|^2 \rho(y^*y)$.

Theorem 2.4.13. [6] *A linear functional ρ on a C^*- algebra $\mathscr{A}$ with the identity I is positive if and only if ρ is bounded and $\|\rho\| = \rho(I)$.*

From Theorem 2.4.13, the following corollary is immediately obtained.

Corollary 2.4.14. [6] *A positive linear functional ρ on a C^*- algebra $\mathscr{A}$ with identity is a state of $\mathscr{A}$ if and only if $\|\rho\| = 1$.*

Theorem 2.4.15. [6] *Suppose that* $\mathscr{A}$ *is a* C^*- *algebra with the identity* I, *and* $x \in \mathscr{A}$.

(1) *If* $\rho(x) = 0$ *for each state* ρ *of* $\mathscr{A}$, *then* $x = 0$.

(2) *If* $\rho(x)$ *is real for each state* ρ *of* $\mathscr{A}$, *then* x *is self-adjoint.*

(3) *If* $\rho(x) \geq 0$ *for each state* ρ *of* $\mathscr{A}$, *then* $x \in \mathscr{A}^+$.

(4) *If* x *is normal, there is a state* ρ *of* $\mathscr{A}$ *such that* $|\rho(x)| = \|x\|$.

Theorem 2.4.16. [6] *Each bounded linear functional on* C^*- *algebra* $\mathscr{A}$ *with identity is a linear combination of at most four states.*

Proposition 2.4.17. [6] *Let* $\mathscr{A}$ *be a* C^*- *algebra with identity. Then the map* $a \mapsto a^*a$ *is continuous.*

Chapter 3

Sequence Spaces of Matrices over C^*-Algebras

In this chapter, we extend some results in [13] to the setting of matrices over commutative C^*-algebras. Throughout this chapter, we let $\mathscr{A}$ be a commutative C^*-algebra with the identity I.

3.1 Classes of matrices over commutative C^*- algebras

In the first section, we study some classes of infinite matrices over the fixed commutative C^*- algebra $\mathscr{A}$ with identity. These classes are the initial spaces for defining some sequence spaces of matrices which will be presented in the next section.

Let $\mathcal{M}^\infty(\mathscr{A})$ be the class of all infinite matrices whose entries are from the C^*-algebra $\mathscr{A}$. It is obvious that the class $\mathcal{M}^\infty(\mathscr{A})$ is a complex vector space under the usual addition and scalar multiplication of infinite matrices over a vector space. For any matrix $A = [a_{jk}]$ in $\mathcal{M}^\infty(\mathscr{A})$ and $f \in \mathscr{A}^*$, let $f(A) = [f(a_{jk})]$. Let

$$\mathcal{M}_b^\infty(\mathscr{A}) = \{A \in \mathcal{M}^\infty(\mathscr{A}) : \varphi(A) \in \mathcal{B}(l_2) \quad \text{for all } \varphi \in \mathscr{S}(\mathscr{A})\}.$$

It is clear that $\mathcal{M}_b^\infty(\mathscr{A})$ is a vector subspace of $\mathcal{M}^\infty(\mathscr{A})$. First, we will define a norm on the vector space $\mathcal{M}_b^\infty(\mathscr{A})$.

Theorem 3.1.1. *For each $A \in \mathcal{M}_b^\infty(\mathscr{A})$, $\sup\limits_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A)\| < \infty$.*

Proof. Let $A = [a_{ji}] \in \mathcal{M}_b^\infty(\mathscr{A})$. Then by the fact that every $f \in \mathscr{A}^*$ is a linear combination of at most four states, we have $f(A) \in \mathcal{B}(l_2)$ for all $f \in \mathscr{A}^*$. Thus the function $L_A : \mathscr{A}^* \to \mathcal{B}(l_2)$ defined by $L_A(f) = f(A)$ for all $f \in \mathscr{A}^*$, is well defined. Since for all $f_1, f_2 \in \mathscr{A}^*$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} L_A(\alpha f_1 + \beta f_2) &= (\alpha f_1 + \beta f_2)(A) = [\alpha f_1(a_{ji}) + \beta f_2(a_{ji})] \\ &= [\alpha f_1(a_{ji})] + [\beta f_2(a_{ji})] = \alpha[f_1(a_{ji})] + \beta[f_2(a_{ji})] \\ &= \alpha L_A(f_1) + \beta L_A(f_2), \end{aligned}$$

it follows that the operator L_A is linear. Next, we will show that the graph $G(L_A) := \{(f, L_A(f)) : f \in \mathscr{A}^*\}$ of the linear operator L_A is closed in the Banach space $\mathscr{A}^* \bigoplus_{l_2} \mathcal{B}(l_2)$. To see this, we let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{A}^*$ which converges to an element f in $\mathscr{A}^*$, and suppose that the sequence $\{L_A(f_n)\}_{n=1}^{\infty}$ converges to a matrix $B = [\beta_{ji}]$ in $\mathcal{B}(l_2)$. We want to show that $B = L_A(f)$. For each $(j,i) \in \mathbb{N} \times \mathbb{N}$, we have $|f_n(a_{ji}) - f(a_{ji})| \leq \|a_{ji}\| \, \|f_n - f\|$ for all n. Thus, by the convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ to f in $\mathscr{A}^*$, we have for each $(j,i) \in \mathbb{N} \times \mathbb{N}$ that the sequence $\{f_n(a_{ji})\}_{n=1}^{\infty}$ converges to $f(a_{ji})$ (in $\mathbb{C}$). As for each $(j,i) \in \mathbb{N} \times \mathbb{N}$, we have $|f_n(a_{ji}) - \beta_{ji}| \leq \|f_n(A) - B\|$ for all n, it follows by the convergence of $\{L_A(f_n)\}_{n=1}^{\infty}$ to B in $\mathcal{B}(l_2)$ that $\{f_n(a_{ji})\}_{n=1}^{\infty}$ also converges to β_{ji} for all $(j,i) \in \mathbb{N} \times \mathbb{N}$. So $f(a_{ji}) = \beta_{ji}$ for all $(j,i) \in \mathbb{N} \times \mathbb{N}$, which implies that $L_A(f) = [f(a_{ji})] = [\beta_{ji}] = B$. Thus the graph $G(L_A)$ is closed in $\mathscr{A}^* \bigoplus_{l_2} \mathcal{B}(l_2)$. Hence, by the closed graph theorem, we have L_A is bounded. It follows that $\sup\limits_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A)\| = \sup\limits_{\varphi \in \mathscr{S}(\mathscr{A})} \|L_A(\varphi)\| \leq \sup\limits_{\|f\| \leq 1} \|L_A(f)\| = \|L_A\| < \infty$. $\square$

From Theorem 3.1.1, a norm on the vector space $\mathcal{M}_b^{\infty}(\mathscr{A})$ can reasonably be defined as follows:

$$\|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} = \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A)\|.$$

Theorem 3.1.2. *The vector space $\mathcal{M}_b^{\infty}(\mathscr{A})$ equipped with this norm $\|\cdot\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}$ is a Banach space.*

Proof. First, we will show that $\|\cdot\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}$ is a norm on $\mathcal{M}_b^{\infty}(\mathscr{A})$. It is clear that $\|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \geq 0$ and $\|\alpha A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} = |\alpha| \, \|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}$ for all $A \in \mathcal{M}_b^{\infty}(\mathscr{A})$ and $\lambda \in \mathbb{C}$. If $A = 0$, then $\varphi(A) = 0$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. Thus $\|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} = 0$. Conversely, if $\|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} = 0$, then $\|\varphi(A)\| = 0$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. It follows that $\varphi(A) = 0$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. Hence each entry of A is mapped by φ to 0 for all $\varphi \in \mathscr{S}(\mathscr{A})$. So all entries of A are 0, which yields $A = 0$. Therefore, for each $A \in \mathcal{M}_b^{\infty}(\mathscr{A})$, we have $\|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} = 0$ if and only if $A = 0$. For any $A, B \in \mathcal{M}_b^{\infty}(\mathscr{A})$, we have

$$\begin{aligned}
\|A + B\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} &= \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A) + \varphi(B)\| \leq \sup_{\varphi \in \mathscr{S}(\mathscr{A})} (\|\varphi(A)\| + \|\varphi(B)\|) \\
&\leq \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A)\| + \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(B)\| \\
&= \|A\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} + \|B\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}.
\end{aligned}$$

So $\|\cdot\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}$ is a norm on $\mathcal{M}_b^{\infty}(\mathscr{A})$. Next, we will show that $\mathcal{M}_b^{\infty}(\mathscr{A})$ endowed with the norm $\|\cdot\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}$ is a Banach space. To prove this, let $\left\{A_n = \left[a_{ji}^{(n)}\right]\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Then there exists $N \in \mathbb{N}$ such that for each $\varphi \in \mathscr{S}(\mathscr{A})$,

$$\|\varphi(A_n) - \varphi(A_m)\| \leq \|A_n - A_m\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \frac{\epsilon}{2} \text{ for all } n, m \geq N \qquad (\star)$$

Thus $\{\varphi(A_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}(l_2)$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. Since $\mathcal{B}(l_2)$ is a Banach space, we have for each $\varphi \in \mathscr{S}(\mathscr{A})$ that there exists $A_\varphi = \left[a_{ji}^{(\varphi)}\right] \in \mathcal{B}(l_2)$

such that $\varphi(A_n) \to A_\varphi$. By $(\star)$ and the normality of elements in the commutative C^*-algebra $\mathscr{A}$, we have for each $(j,i) \in \mathbb{N} \times \mathbb{N}$ that

$$\begin{aligned} \left\| a_{ji}^{(n)} - a_{ji}^{(m)} \right\| &= \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \left| \varphi\left(a_{ji}^{(n)} - a_{ji}^{(m)} \right) \right| \leq \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \| \varphi(A_n - A_m) \| \\ &< \epsilon \text{ for all } m, n \geq N. \end{aligned}$$

This implies that $\left\{ a_{ji}^{(n)} \right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathscr{A}$ for all (j,i). Hence, by the completeness of $\mathscr{A}$, we have, for each (j,i), an element $a_{ji} \in \mathscr{A}$ such that $a_{ji}^{(n)} \to a_{ji}$. Let $A = [a_{ji}]$. We claim that $A \in \mathcal{M}_b^\infty(\mathscr{A})$ and $A_n \to A$. To see that $A \in \mathcal{M}_b^\infty(\mathscr{A})$, let $\varphi \in \mathscr{S}(\mathscr{A})$. Since φ is continuous, we have $\varphi\left(a_{ji}^{(n)}\right) \to \varphi(a_{ji})$ for all (j,i). As $\varphi(A_n) \to A_\varphi$, we have for each (j,i) that $\varphi\left(a_{ji}^{(n)}\right) \to a_{ji}^{(\varphi)}$. So $a_{ji}^{(\varphi)} = \varphi(a_{ji})$ for all (j,i), which implies that $A_\varphi = \varphi(A)$. Thus $\varphi(A) \in \mathcal{B}(l_2)$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. It follows that $A \in \mathcal{M}_b^\infty(\mathscr{A})$ as claimed. We now have $\varphi(A_n) \to \varphi(A)$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. Thus, by taking the limits as $m \to \infty$ on both sides of $(*)$, we have for each $\varphi \in \mathscr{S}(\mathscr{A})$ that

$$\|\varphi(A_n) - \varphi(A)\| \leq \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Hence

$$\|A_n - A_m\|_{\mathcal{M}_b^\infty(\mathscr{A})} = \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A_n) - \varphi(A)\| < \epsilon \text{ for all } n \geq N.$$

It follows that $\{A_n\}_{n=1}^\infty$ converges to A. Therefore, $\mathcal{M}_b^\infty(\mathscr{A})$ is a Banach space. $\square$

We do not however know if the Banach space $\mathcal{M}_b^\infty(\mathscr{A})$ is closed under the Schur product multiplication. Another interesting subclass of $\mathcal{M}^\infty(\mathscr{A})$ is the class of compact-like matrices defined below. Let

$$\mathcal{K}(\mathscr{A}) = \left\{ A \in \mathcal{M}_b^\infty(\mathscr{A}) : \|A - A_{n\lrcorner}\|_{\mathcal{M}_b^\infty(\mathscr{A})} \to 0 \right\}.$$

For the case where $\mathscr{A} = \mathbb{C}$, the set $\mathcal{K}(\mathbb{C})$ is exactly the set of all compact matrices regarded as operators on l_2 and will be denoted by just $\mathcal{K}$. We obtain some characterizations of the set $\mathcal{K}(\mathscr{A})$ as follows.

Theorem 3.1.3. *Let $A \in \mathcal{M}_b^\infty(\mathscr{A})$. Then the following are equivalent.*

(1) *The matrix A belongs to $\mathcal{K}(\mathscr{A})$.*

(2) *There is a sequence $\{F_n\}_{n=1}^\infty$ of matrices in $\mathcal{M}^\infty(\mathscr{A})$ with finitely many non-zero entries such that $\|F_n - A\|_{\mathcal{M}_b^\infty(\mathscr{A})} \to 0$.*

(3) *The matrix $\varphi(A)$ belongs to $\mathcal{K}$ for all $\varphi \in \mathscr{S}(\mathscr{A})$ and the map $\varphi \mapsto \varphi(A)$ from $\mathscr{S}(\mathscr{A})$ equipped with the topology relative to the weak* topology on $\mathscr{A}^*$ into $\mathcal{K}$ is continuous.*

Proof. (1) $\Rightarrow$ (2). By the definition of $\mathcal{K}(\mathscr{A})$, the implication (1) $\Rightarrow$ (2) is immediately obtained.

(2) $\Rightarrow$ (3). Suppose that there is a sequence $\{F_n\}_{n=1}^{\infty}$ of matrices in $\mathcal{M}^{\infty}(\mathscr{A})$ with finitely many non-zero entries such that $\|\!|F_n - A|\!\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \to 0$. Since for each $\varphi \in \mathscr{S}(\mathscr{A})$, we have $\|\varphi(A) - \varphi(F_n)\| \leq \|\!|A - F_n|\!\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}$ for all n, it follows by the assumption that $\|\varphi(A) - \varphi(F_n)\| \to 0$. Since for each $\varphi \in \mathscr{S}(\mathscr{A})$ and positive integer n, the matrix $\varphi(F_n)$ is finite rank, we have it is compact. Thus, by the closedness of $\mathcal{K}$ in $\mathcal{B}(l_2)$, we have $\varphi(A) \in \mathcal{K}$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. Hence the map $\varphi \mapsto \varphi(A)$ from $\mathscr{S}(\mathscr{A})$ into $\mathcal{K}$ is well defined. To see that it is continuous, where $\mathscr{S}(\mathscr{A})$ is equipped with the topology relative to the weak* topology on $\mathscr{A}^*$, let $\epsilon > 0$ be given, and suppose that $\varphi_\alpha \to^{w^*} \varphi$ in $\mathscr{S}(\mathscr{A})$. Then by the assumption, there exists a positive integer N such that $\|\!|A - F_N|\!\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \dfrac{\epsilon}{3}$. Assume that $F_N = [b_{ji}]$. Then there is a positive integer ν such that $b_{ji} = 0$ for all $j, i > \nu$. Since $\varphi_\alpha \to^{w^*} \varphi$, there exists α_0 such that for each $1 \leq j, i \leq \nu$, $|\varphi_\alpha(b_{ji}) - \varphi(b_{ji})| < \dfrac{\epsilon}{3\nu^{3/2}}$ for all $\alpha \succeq \alpha_0$. It follows that

$$\begin{aligned}
\|\varphi_\alpha(A) - \varphi(A)\| &\leq \|\varphi_\alpha(A) - \varphi_\alpha(F_N)\| + \|\varphi(A) - \varphi(F_N)\| \\
&\quad + \|\varphi(F_N) - \varphi_\alpha(F_N)\| \\
&\leq \|\varphi_\alpha(A) - \varphi_\alpha(F_N)\| + \|\varphi(A) - \varphi(F_N)\| \\
&\quad + \left\{ \sum_{j=1}^{\nu} \left(\sum_{i=1}^{\nu} |\varphi_\alpha(b_{ji}) - \varphi(b_{ji})| \right)^2 \right\}^{1/2} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ for all } \alpha \succeq \alpha_0.
\end{aligned}$$

Thus $\varphi_\alpha(A) \to \varphi(A)$. Accordingly, the map $\varphi \mapsto \varphi(A)$ from $\mathscr{S}(\mathscr{A})$ equipped with the topology relative to the weak* topology on $\mathscr{A}^*$ into $\mathcal{K}$, is continuous.

(3) $\Rightarrow$ (1). Suppose that the assertion (3) holds. To prove that A belongs to $\mathcal{K}(\mathscr{A})$, let $\epsilon > 0$. Then by the continuity of the map $\varphi \mapsto \varphi(A)$, we have for each $\varphi \in \mathscr{S}(\mathscr{A})$ that there exists an open set V_φ in $\mathscr{S}(\mathscr{A})$ such that $\varphi \in V_\varphi$ and $\|\varphi(A) - \rho(A)\| < \dfrac{\epsilon}{4}$ for all $\rho \in V_\varphi$. We now have that the family $\{V_\varphi : \varphi \in \mathscr{S}(\mathscr{A})\}$ is an open cover of $\mathscr{S}(\mathscr{A})$. Thus, by the compactness of $\mathscr{S}(\mathscr{A})$, there exist $\varphi_1, \varphi_2, ..., \varphi_k \in \mathscr{S}(\mathscr{A})$ such that $\mathscr{S}(\mathscr{A}) = V_{\varphi_1} \cup V_{\varphi_2} \cup ... \cup V_{\varphi_k}$. By the compactness of matrices $\varphi_1(A), \varphi_2(A), ..., \varphi_k(A)$, there is a positive integer N such that for each $\mu = 1, 2, ..., k$, $\|\varphi_\mu(A) - \varphi_\mu(A_{n_\lrcorner})\| < \dfrac{\epsilon}{4}$ for all $n \geq N$. Next, let $\varphi \in \mathscr{S}(\mathscr{A})$. Then there is $1 \leq \nu \leq k$ such that $\varphi \in V_{\varphi_\nu}$. Hence

$$\begin{aligned}
\|\varphi(A) - \varphi(A_{n_\lrcorner})\| &\leq \|\varphi(A) - \varphi_\nu(A)\| + \|\varphi_\nu(A_{n_\lrcorner}) - \varphi(A_{n_\lrcorner})\| \\
&\quad + \|\varphi_\nu(A) - \varphi_\nu(A_{n_\lrcorner})\| \\
&= \|\varphi(A) - \varphi_\nu(A)\| + \|(\varphi_\nu(A) - \varphi(A))_{n_\lrcorner}\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|\varphi_\nu(A) - \varphi_\nu(A_{n_\lrcorner})\| \\
&\leq \; 2\|\varphi(A) - \varphi_\nu(A)\| + \|\varphi_\nu(A) - \varphi_\nu(A_{n_\lrcorner})\| \\
&< \; \frac{2\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{4} \text{ for all } n \geq N.
\end{aligned}
$$

Since φ in $\mathscr{S}(\mathscr{A})$ was given arbitrarily, we have by taking the supremun over all φ in $\mathscr{S}(\mathscr{A})$ that $\|\|A - A_{n_\lrcorner}\|\|_{\mathcal{M}_b^\infty(\mathscr{A})} < \epsilon$ for all $n \geq N$. This yields the membership of A in $\mathcal{K}(\mathscr{A})$. The proof is complete. □

From the theorem above, we have that the set $\mathcal{K}(\mathscr{A})$ is exactly the closure, in $\mathcal{M}_b^\infty(\mathscr{A})$, of the set of all matrices over $\mathscr{A}$ having finitely many non-zero entries. Thus the following corollary is immediately obtained.

Corollary 3.1.4. *The set $\mathcal{K}(\mathscr{A})$ is a Banach subspace of $\mathcal{M}_b^\infty(\mathscr{A})$.*

For any $A = [a_{ji}] \in \mathcal{M}^\infty(\mathscr{A})$, let $A^{[2]} = [a_{ji}^* a_{ji}]$. Let

$$\mathcal{S}^2(\mathscr{A}) = \left\{ A \in \mathcal{M}^\infty(\mathscr{A}) : A^{[2]} \in \mathcal{M}_b^\infty(\mathscr{A}) \right\}.$$

The set $\mathcal{S}^2(\mathscr{A})$ was first defined in [4] by P. Chaisuriya, S.-C. Ong, and S.-W. Wang over any C^*-algebra with identity. For completeness of the contents in this thesis, we will study the set $\mathcal{S}^2(\mathscr{A})$ restricted to be over just a commutative C^*-algebra with identity again here.

We define a norm on $\mathcal{S}^2(\mathscr{A})$ by

$$\|\|A\|\|_2 = \left\|\left\| A^{[2]} \right\|\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2}.$$

The following Cauchy-Schwarz-type inequality and Minkowsky-type inequality were first established in [3] by P. Chaisuriya and S.-C. Ong.

Theorem 3.1.5. (Cauchy-Schwarz-type inequality) *For any infinite scalar matrices A and B, $\|\|A \bullet B\|\| \leq \left\| A^{[2]} \right\|^{1/2} \left\| B^{[2]} \right\|^{1/2}$.*

Theorem 3.1.6. (Minkowsky-type inequality) *For any infinite scalar matrices A and B, $\left\| (A+B)^{[2]} \right\|^{1/2} \leq \left\| A^{[2]} \right\|^{1/2} + \left\| B^{[2]} \right\|^{1/2}$.*

The following lemma, which is very useful for the research, is a consequence of the above Cauchy-Schwarz-type inequality.

Lemma 3.1.7. *For any $A, B \in \mathcal{S}^2(\mathscr{A})$,*

$$\left\|\left\| A^{[2]} - B^{[2]} \right\|\right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \leq \left(\|\|A\|\|_2 + \|\|B\|\|_2 \right) \|\|A - B\|\|_2 .$$

Proof. Let $A = [a_{ji}], B = [b_{ji}] \in \mathcal{S}^2(\mathscr{A})$ and $\varphi \in \mathscr{S}(\mathscr{A})$. Then by Proposition 2.4.12(1), we have for each $(j, i) \in \mathbb{N} \times \mathbb{N}$ that

$$\left| \varphi\left(a_{ji}^* a_{ji} - b_{ji}^* b_{ji} \right) \right| \;=\; \left| \varphi\left(\left(a_{ji}^* a_{ji} - a_{ji}^* b_{ji} \right) + \left(a_{ji}^* b_{ji} - b_{ji}^* b_{ji} \right) \right) \right|$$

$$
\begin{aligned}
&= \left|\varphi\left(a_{ji}^*(a_{ji}-b_{ji})+(a_{ji}-b_{ji})^*b_{ji}\right)\right| \\
&\leq \left|\varphi\left(a_{ji}^*(a_{ji}-b_{ji})\right)\right| + \left|\varphi\left((a_{ji}-b_{ji})^*b_{ji}\right)\right| \\
&\leq \varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2} \\
&\qquad +\varphi\left((a_{ji}-b_{ji})(a_{ji}-b_{ji})^*\right)^{1/2}\varphi\left(b_{ji}^*b_{ji}\right)^{1/2} \\
&= \varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2} \\
&\qquad +\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2}\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}.
\end{aligned}
$$

So, by the Cauchy-Schwarz-type inequality above, we obtain

$$
\begin{aligned}
\left\|\varphi\left(A^{[2]}\right)-\varphi\left(B^{[2]}\right)\right\| &\leq \left\|\left[\left|\varphi\left(|a_{ji}|^2-|b_{ji}|^2\right)\right|\right]\right\| \\
&\leq \left\|\left[\varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\right]\bullet\left[\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2}\right]\right\| \\
&\qquad +\left\|\left[\varphi\left((a_{ji}-b_{ji})(a_{ji}-b_{ji})^*\right)^{1/2}\right]\bullet\left[\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}\right]\right\| \\
&= \left\|\left[\varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\right]\bullet\left[\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2}\right]\right\| \\
&\qquad +\left\|\left[\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2}\right]\bullet\left[\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}\right]\right\| \\
&\leq \left\|\left[\varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\right]^{[2]}\right\|^{1/2}\left\|\left[\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2}\right]^{[2]}\right\|^{1/2} \\
&\qquad +\left\|\left[\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)^{1/2}\right]^{[2]}\right\|^{1/2}\left\|\left[\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}\right]^{[2]}\right\|^{1/2} \\
&\leq \left(\left\|\left[\varphi\left(a_{ji}^*a_{ji}\right)\right]\right\|^{1/2}+\left\|\left[\varphi\left(b_{ji}^*b_{ji}\right)\right]\right\|^{1/2}\right)\left\|\left[\varphi\left((a_{ji}-b_{ji})^*(a_{ji}-b_{ji})\right)\right]\right\|^{1/2} \\
&\leq \left(\|A\|_2+\|B\|_2\right)\|A-B\|_2.
\end{aligned}
$$

Thus, by taking the supremun over all $\varphi\in\mathscr{S}(\mathscr{A})$, we obtain the inequality

$$\left\|A^{[2]}-B^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}\leq\left(\|A\|_2+\|B\|_2\right)\|A-B\|_2$$

as asserted. The proof is complete. □

We are now ready for proving the completeness of the set $\mathcal{S}^2(\mathscr{A})$.

Theorem 3.1.8. *The set $\mathcal{S}^2(\mathscr{A})$ endowed with the norm $\|\cdot\|_2$ is a Banach algebra under the Schur product multiplication.*

Proof. We will show first that $\|\cdot\|_2$ is a norm on $\mathcal{S}^2(\mathscr{A})$. Let $A,B\in\mathcal{S}^2(\mathscr{A})$ and $\alpha\in\mathbb{C}$. Then

$$
\begin{aligned}
\left\|\varphi\left((\alpha A)^{[2]}\right)\right\|^{1/2} &= \left\|\left[\varphi\left(\overline{\alpha}a_{ji}^*\alpha a_{ji}\right)\right]\right\|^{1/2}=\left\||\alpha|^2\left[\varphi\left(a_{ji}^*a_{ji}\right)\right]\right\|^{1/2} \\
&= |\alpha|\left\|\varphi\left(A^{[2]}\right)\right\|^{1/2}\quad\text{for all }\varphi\in\mathscr{S}(\mathscr{A}).
\end{aligned}
$$

By taking suprema, as φ runs over the set $\mathscr{S}(\mathscr{A})$, on both sides of the equation, we get $\|\alpha A\|_2 = |\alpha| \, \|A\|_2$. Next, we will show that $\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$. By Proposition 2.4.12(2), and the Mikowsky-type inequality, we have for any φ in $\mathscr{S}(\mathscr{A})$ that

$$\begin{aligned}
\left\|\varphi\left((A+B)^{[2]}\right)\right\|^{1/2} &= \left\|\left[\varphi\left((a_{ji}+b_{ji})^*\left(a_{ji}+b_{ji}\right)\right)\right]\right\|^{1/2} \\
&\leq \left\|\left[\left(\varphi\left(a_{ji}^*a_{ji}\right)^{1/2}+\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}\right)^2\right]\right\|^{1/2} \\
&= \left\|\left(\left[\varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\right]+\left[\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}\right]\right)^{[2]}\right\|^{1/2} \\
&\leq \left\|\left[\varphi\left(a_{ji}^*a_{ji}\right)^{1/2}\right]^{[2]}\right\|^{1/2} + \left\|\left[\varphi\left(b_{ji}^*b_{ji}\right)^{1/2}\right]^{[2]}\right\|^{1/2} \\
&= \left\|\varphi\left((A)^{[2]}\right)\right\|^{1/2} + \left\|\varphi\left((B)^{[2]}\right)\right\|^{1/2}.
\end{aligned}$$

This implies that $A+B \in \mathcal{S}^2(\mathscr{A})$ and

$$\|A+B\|_2 = \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})} \left\|\varphi\left((A+B)^{[2]}\right)\right\|\right)^{1/2} \leq \|A\|_2 + \|B\|_2 .$$

Thus $\mathcal{S}^2(\mathscr{A})$ is a vector space and $\|\cdot\|_2$ is a norm on $\mathcal{S}^2(\mathscr{A})$. Next, we will show that $\mathcal{S}^2(\mathscr{A})$ is a normed algebra under the Schur product multiplication. By Proposition 2.4.12(3), we have

$$\begin{aligned}
\left\|\varphi\left((A \bullet B)^{[2]}\right)\right\| &= \left\|\left[\varphi\left((a_{ji}b_{ji})^*(a_{ji}b_{ji})\right)\right]\right\| \leq \left\|\left[\|a_{ji}\|^2\varphi\left(b_{ji}^*b_{ji}\right)\right]\right\| \\
&= \left\|\left[\left\|a_{ji}^*a_{ji}\right\|\varphi\left(b_{ji}^*b_{ji}\right)\right]\right\| = \left\|\sup_{\phi\in\mathscr{S}(\mathscr{A})}\phi\left(a_{ji}^*a_{ji}\right)\left[\varphi\left(b_{ji}^*b_{ji}\right)\right]\right\| \\
&\leq \sup_{\phi\in\mathscr{S}(\mathscr{A})}\left\|\phi\left(A^{[2]}\right)\right\|\left\|\left[\varphi\left(b_{ji}^*b_{ji}\right)\right]\right\| = \|A\|_2^2\left\|\varphi(B^{[2]})\right\| \\
&\leq \|A\|_2^2\,\|B\|_2^2 \qquad \text{for all } \varphi\in\mathscr{S}(\mathscr{A}).
\end{aligned}$$

This yields $A \bullet B \in \mathcal{S}^2(\mathscr{A})$. Moreover, $\|A \bullet B\|_2 \leq \|A\|_2 \|B\|_2$. Thus $\mathcal{S}^2(\mathscr{A})$ is a normed algebra under the Schur product. Next, we will show that $\mathcal{S}^2(\mathscr{A})$ is a Banach space. Let $\left\{A_n = \left[a_{ji}^{(n)}\right]\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{S}^2(\mathscr{A})$. Since for each $(j,i) \in \mathbb{N}\times\mathbb{N}$,

$$\begin{aligned}
\left\|a_{ji}^{(m)} - a_{ji}^{(n)}\right\| &= \left\|\left(a_{ji}^{(m)} - a_{ji}^{(n)}\right)^*\left(a_{ji}^{(m)} - a_{ji}^{(n)}\right)\right\|^{1/2} \\
&= \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\varphi\left(\left(a_{ji}^{(m)} - a_{ji}^{(n)}\right)^*\left(a_{ji}^{(m)} - a_{ji}^{(n)}\right)\right)\right)^{1/2}
\end{aligned}$$

$$\leq \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})} \left\| \left[\varphi\left(\left(a_{ji}^{(m)} - a_{ji}^{(n)}\right)^* \left(a_{ji}^{(m)} - a_{ji}^{(n)}\right)\right)\right] \right\| \right)^{1/2}$$

$$= \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})} \left\| \varphi\left(A_n - A_m\right)^{[2]} \right\| \right)^{1/2}$$

$$= \left\|\left| A_n - A_m \right\|\right|_2 \ \text{ for all } m, n,$$

we have $\left\{a_{ji}^{(n)}\right\}_{n=1}^{\infty}$ is a cauchy sequence in $\mathscr{A}$ for all (j,i). By the completeness of $\mathscr{A}$, we have for each (j,i) that there exists an $a_{ji} \in \mathscr{A}$ such that $a_{ji}^{(n)} \to a_{ji}$ in $\mathscr{A}$. Let $A = [a_{ji}]$. We will show that $A \in \mathcal{S}^2(\mathscr{A})$ and $A_n \to A$. To see that $A \in \mathcal{S}^2(\mathscr{A})$, let M be a positive real number such that $|||A_n|||_2 \leq M$ for all $n \in \mathbb{N}$. Notice that the existence of such an M is from the boundedness of the Cauchy sequence $\{A_n\}_{n=1}^{\infty}$. By Lemma 3.1.7, we have

$$\left|\left|\left| A_n^{[2]} - A_m^{[2]} \right|\right|\right|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \leq 2M \, |||A_n - A_m|||_2 \ \text{ for all } n, m.$$

This implies that $\left\{A_n^{[2]}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Hence, by the completeness of $\mathcal{M}_b^{\infty}(\mathscr{A})$, there is a matrix $B = [b_{ji}] \in \mathcal{M}_b^{\infty}(\mathscr{A})$ such that $A_n^{[2]} \to B$. From this, we obtain for each (j,i) that $a_{ji}^{(n)^*} a_{ji}^{(n)} \to b_{ji}$. Since $a_{ji}^{(n)} \to a_{ji}$, we have by the continuity of the map $a \mapsto a^*a$ on $\mathscr{A}$ that $a_{ji}^{(n)^*} a_{ji}^{(n)} \to a_{ji}^* a_{ji}$ for all (j,i). Accordingly, $a_{ji}^* a_{ji} = b_{ji}$ for all (j,i), which yields $A^{[2]} = B$. Therefore, $A \in \mathcal{S}^2(\mathscr{A})$ as required. Finally, we will show that $A_n \to A$. To this end, let $\epsilon > 0$ be given. Then there is a positive integer N such that

$$|||A_n - A_m|||_2 < \frac{\epsilon}{2} \ \text{ for all } m, n \geq N.$$

Let $\mathbf{x} = \{\xi_i\}_{i=1}^{\infty} \in l_2$ with $\|\mathbf{x}\|_2 \leq 1$, and let $\varphi \in \mathscr{S}(\mathscr{A})$. Then for each fixed pair (ν, μ) of positive integers, we have

$$\left\{ \sum_{j=1}^{\nu} \left(\sum_{i=1}^{\mu} \varphi\left(\left(a_{ji}^{(n)} - a_{ji}^{(m)}\right)^* \left(a_{ji}^{(n)} - a_{ji}^{(m)}\right)\right) |\xi_i| \right)^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \varphi\left(\left(a_{ji}^{(n)} - a_{ji}^{(m)}\right)^* \left(a_{ji}^{(n)} - a_{ji}^{(m)}\right)\right) |\xi_i| \right)^2 \right\}^{1/2}$$

$$= \left\| \varphi\left((A_n - A_m)^{[2]} \right) |\mathbf{x}| \right\|_2 \leq \left\| \varphi\left((A_n - A_m)^{[2]} \right) \right\|$$

$$\leq |||A_n - A_m|||_2^2 < \left(\frac{\epsilon}{2}\right)^2 \ \text{ for all } m, n \geq N,$$

where $|\mathbf{x}| = \{|\xi_i|\}_{i=1}^{\infty}$. For any (j,i), since $a_{ji}^{(m)} \to a_{ji}$, we have for each fixed n that $a_{ji}^{(n)} - a_{ji}^{(m)} \to a_{ji}^{(n)} - a_{ji}$. Thus, by the continuity of the map $a \mapsto a^*a$ on

$\mathscr{A}$, $\left(a_{ji}^{(n)} - a_{ji}^{(m)}\right)^* \left(a_{ji}^{(n)} - a_{ji}^{(m)}\right) \to \left(a_{ji}^{(n)} - a_{ji}\right)^* \left(a_{ji}^{(n)} - a_{ji}\right)$ for all n. Hence, by taking the limits as $m \to \infty$ on both sides of the above inequality, we obtain for each for pair (ν, μ) of positive integers that

$$\left\{\sum_{j=1}^{\nu}\left(\sum_{i=1}^{\mu}\varphi\left(\left(a_{ji}^{(n)} - a_{ji}\right)^* \left(a_{ji}^{(n)} - a_{ji}\right)\right)|\xi_i|\right)^2\right\}^{1/2} \leq \left(\frac{\epsilon}{2}\right)^2 \text{ for all } n \geq N.$$

Taking the limit as $\mu \to \infty$ and then as $\nu \to \infty$ on both sides of the latest inequality above, we obtain

$$\begin{aligned}
\left\|\varphi\left((A_n - A)^{[2]}\right)\mathbf{x}\right\|_2 &\leq \left\|\varphi\left((A_n - A)^{[2]}\right)|\mathbf{x}|\right\|_2 \\
&= \left\{\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\varphi\left(\left(a_{ji}^{(n)} - a_{ji}\right)^* \left(a_{ji}^{(n)} - a_{ji}\right)\right)|\xi_i|\right)^2\right\}^{1/2} \\
&\leq \left(\frac{\epsilon}{2}\right)^2 \text{ for all } n \geq N.
\end{aligned}$$

It follows that

$$\left\|\varphi\left((A_n - A)^{[2]}\right)\right\| \leq \left(\frac{\epsilon}{2}\right)^2 \text{ for all } n \geq N.$$

Since $\varphi \in \mathscr{S}(\mathscr{A})$ was given arbitrarily, we have

$$\begin{aligned}
|||A_n - A|||_2 &= \left|\left|\left|(A_n - A)^{[2]}\right|\right|\right|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2} = \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \left\|\varphi\left((A_n - A)^{[2]}\right)\right\|^{1/2} \\
&\leq \frac{\epsilon}{2} \text{ for all } n \geq N.
\end{aligned}$$

Consequently, $A_n \to A$. The proof is complete. □

Proposition 3.1.9. *The map $A \mapsto A^{[2]}$ from $\mathcal{S}^2(\mathscr{A})$ into $\mathcal{M}_b^\infty(\mathscr{A})$ is continuous.*

Proof. Suppose that $\{A_n\}_{n=1}^\infty$ is a sequence in $\mathcal{S}^2(\mathscr{A})$ that converges to A in $\mathcal{S}^2(\mathscr{A})$. We will show that $\left\{A_n^{[2]}\right\}_{n=1}^\infty$ converges to $A^{[2]}$ in $\mathcal{M}_b^\infty(\mathscr{A})$. Since $\{A_n\}_{n=1}^\infty$ converges to A, there exits $M > 0$ such that $|||A||| \leq M$ and $|||A_n||| \leq M$ for all $n \in \mathbb{N}$. So, by Lemma 3.1.7,

$$\left|\left|\left|A_n^{[2]} - A^{[2]}\right|\right|\right|_{\mathcal{M}_b^\infty(\mathscr{A})} \leq 2M\,|||A_n - A|||\,.$$

Thus, by taking limits as $n \to \infty$ on both sides of the inequality, we have that $\left|\left|\left|A_n^{[2]} - A^{[2]}\right|\right|\right|_{\mathcal{M}_b^\infty(\mathscr{A})} \to 0$, which yields $\left\{A_n^{[2]}\right\}_{n=1}^\infty$ converges to $A^{[2]}$ in $\mathcal{M}_b^\infty(\mathscr{A})$. Therefore, the map $A \mapsto A^{[2]}$ is continuous. □

We end this section with a Cauchy-Schwarz-type inequality for the norm $|||\cdot|||_{\mathcal{M}_b^\infty(\mathscr{A})}$.

Theorem 3.1.10. *If $A, B \in \mathcal{S}^2(\mathscr{A})$, then*

$$\|A \bullet B\|_{\mathcal{M}_b^\infty(\mathscr{A})} \leq \left\|A^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2} \left\|B^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2}.$$

Proof. Let $A = [a_{ji}]$ and $B = [b_{ji}]$ be members of $\mathcal{S}^2(\mathscr{A})$. Then by Proposition 2.4.12(1), and the Cauchy-Schwarz-type inequality, we have for all $\varphi \in \mathscr{S}(\mathscr{A})$ that

$$\begin{aligned}
\|\varphi(A \bullet B)\| &= \|[\varphi(a_{ji}b_{ji})]\| \leq \left\|[\varphi(a_{ji}^*a_{ji})^{1/2}\varphi(b_{ji}^*b_{ji})^{1/2}]\right\| \\
&= \left\|[\varphi(a_{ji}^*a_{ji})^{1/2}] \bullet [\varphi(b_{ji}^*b_{ji})^{1/2}]\right\| \\
&\leq \left\|[\varphi(a_{ji}^*a_{ji})]\right\|^{1/2} \left\|[\varphi(b_{ji}^*b_{ji})]\right\|^{1/2} \\
&= \left\|\varphi\left(A^{[2]}\right)\right\|^{1/2} \left\|\varphi\left(B^{[2]}\right)\right\|^{1/2} \\
&\leq \left\|A^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2} \left\|B^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2}.
\end{aligned}$$

Thus $A \bullet B$ belongs to $\mathcal{M}_b^\infty(\mathscr{A})$ and we then obtain the inequality $\|A \bullet B\|_{\mathcal{M}_b^\infty(\mathscr{A})} \leq \left\|A^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2} \left\|B^{[2]}\right\|_{\mathcal{M}_b^\infty(\mathscr{A})}^{1/2}$ as asserted. □

3.2 Sequence spaces of matrices in $\mathcal{S}^2(\mathscr{A})$

From the structure of the Banach algebra $\mathcal{S}^2(\mathscr{A})$, we can reasonably define two sets of sequences of matrices over the C^*-algebra $\mathscr{A}$ as follows:

$$\mathscr{O}_b(\mathscr{A}) = \left\{\{A_k\}_{k=1}^\infty \subseteq \mathcal{S}^2(\mathscr{A}) : \left\{\sum_{k=1}^n A_k^{[2]}\right\}_{n=1}^\infty \text{ is bounded in } \mathcal{M}_b^\infty(\mathscr{A})\right\};$$

and

$$\mathscr{O}_c(\mathscr{A}) = \left\{\{A_k\}_{k=1}^\infty \subseteq \mathcal{S}^2(\mathscr{A}) : \left\{\sum_{k=1}^n A_k^{[2]}\right\}_{n=1}^\infty \text{ converges in } \mathcal{M}_b^\infty(\mathscr{A})\right\}.$$

It is clear that $\mathscr{O}_c(\mathscr{A}) \subseteq \mathscr{O}_b(\mathscr{A})$.

Lemma 3.2.1. *Let $\left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^\infty$ be a sequence in $\mathcal{B}(l_2)$ with $a_{ji}^{(k)} \geq 0$ for all i, j, k.*

(1) *The sequence $\left\{\sum_{k=1}^n \left[a_{ji}^{(k)}\right]\right\}_{n=1}^\infty$ is bounded if and only if $\left[\sum_{k=1}^\infty a_{ji}^{(k)}\right] \in \mathcal{B}(l_2)$.*

(2) *If $\left\{\sum_{k=1}^n \left[a_{ji}^{(k)}\right]\right\}_{n=1}^\infty$ is bounded, then $\sup_n \left\|\sum_{k=1}^n \left[a_{ji}^{(k)}\right]\right\| = \left\|\left[\sum_{k=1}^\infty a_{ji}^{(k)}\right]\right\|$.*

Proof. (1) Let $\left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^\infty$ be a sequence in $\mathcal{B}(l_2)$ with $a_{ji}^{(k)} \geq 0$ for all i, j, k. Suppose that the sequence $\left\{\sum_{k=1}^n \left[a_{ji}^{(k)}\right]\right\}_{n=1}^\infty$ is bounded. Then there exists $M > 0$

such that for each $(j,i) \in \mathbb{N} \times \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} a_{ji}^{(k)} \leq \left\| \left[\sum_{k=1}^{n} a_{ji}^{(k)} \right] \right\| = \left\| \sum_{k=1}^{n} \left[a_{ji}^{(k)} \right] \right\| \leq M$ for all n. This implies that the series $\displaystyle\sum_{k=1}^{\infty} a_{ji}^{(k)}$ converges for all (j,i). Next, let $A = \left[\displaystyle\sum_{k=1}^{\infty} a_{ji}^{(k)} \right]$, $M = \sup\limits_{K} \left\| \displaystyle\sum_{k=1}^{K} \left[a_{ji}^{(k)} \right] \right\|$, and let $\mathbf{x} = \{\xi_i\}_{i=1}^{\infty} \in l_2$ with $\|\mathbf{x}\|_2 \leq 1$. Then we have for each n by Minkowski's inequality for scalar sequences that

$$
\begin{aligned}
\|A_{n\lrcorner}\mathbf{x}\|_2 &= \left\{ \sum_{j=1}^{n} \left| \sum_{i=1}^{n} \left(\sum_{k=1}^{\infty} a_{ji}^{(k)} \right) \xi_i \right|^2 \right\}^{1/2} \\
&\leq \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ji}^{(k)} \xi_i \right| \right)^2 \right\}^{1/2} \\
&\leq \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ji}^{(k)} - \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| + \left| \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| \right)^2 \right\}^{1/2} \\
&\leq \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ji}^{(k)} - \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| \right)^2 \right\}^{1/2} \\
&\quad + \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| \right)^2 \right\}^{1/2} \\
&\leq \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ji}^{(k)} - \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| \right)^2 \right\}^{1/2} + \left\| \left(\sum_{k=1}^{K} \left[a_{ji}^{(k)} \right] \right)_{n\lrcorner} \right\| \\
&\leq \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ji}^{(k)} - \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| \right)^2 \right\}^{1/2} + \left\| \left(\sum_{k=1}^{K} \left[a_{ji}^{(k)} \right] \right) \right\| \\
&\leq \left\{ \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ji}^{(k)} - \sum_{k=1}^{K} a_{ji}^{(k)} \right| |\xi_i| \right)^2 \right\}^{1/2} + M \quad \text{for all } K.
\end{aligned}
$$

Hence, by taking the limit as $K \to \infty$, we have for each n that $\|A_{n\lrcorner}\mathbf{x}\|_2 \leq M$ for all $\mathbf{x} \in l_2$ with $\|\mathbf{x}\|_2 \leq 1$. So $\|A_{n\lrcorner}\| \leq M$ for all n. Consequently, $A \in \mathcal{B}(l_2)$. Conversely, suppose that $\left[\displaystyle\sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \in \mathcal{B}(l_2)$. Then $\left\| \left[\displaystyle\sum_{k=1}^{n} a_{ji}^{(k)} \right] \right\| \leq \left\| \left[\displaystyle\sum_{k=1}^{\infty} a_{ji}^{(k)} \right] \right\|$ for all n, which implies that $\left\{\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\}_{n=1}^{\infty}$ is bounded.

(2) Suppose that the sequence $\left\{\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\}_{n=1}^{\infty}$ is bounded. It is clear that

$$\sup_{n}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\| = \sup_{n}\left\|\left[\sum_{k=1}^{n}a_{ji}^{(k)}\right]\right\| \leq \left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)}\right]\right\|.$$

Let $\mathbf{x} = \{\xi_i\}_{i=1}^{\infty} \in l_2$ with $\|\mathbf{x}\|_2 \leq 2$. Then for each positive integer ν,

$$\begin{aligned}
\left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)}\right]_{\nu\lrcorner}\mathbf{x}\right\|_2 &= \left(\sum_{j=1}^{\nu}\left|\sum_{i=1}^{\nu}\sum_{k=1}^{\infty}a_{ji}^{(k)}\xi_i\right|^2\right)^{1/2} \\
&\leq \left\{\sum_{j=1}^{\nu}\left(\sum_{i=1}^{\nu}\sum_{k=1}^{\infty}a_{ji}^{(k)}|\xi_i|\right)^2\right\}^{1/2} \\
&= \lim_{n\to\infty}\left\{\sum_{j=1}^{\nu}\left(\sum_{i=1}^{\nu}\sum_{k=1}^{n}a_{ji}^{(k)}|\xi_i|\right)^2\right\}^{1/2} \\
&\leq \lim_{n\to\infty}\left\{\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\sum_{k=1}^{n}a_{ji}^{(k)}|\xi_i|\right)^2\right\}^{1/2} \\
&= \lim_{n\to\infty}\left\|\left(\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right)|\mathbf{x}|\right\|_2 \leq \lim_{n\to\infty}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\|.
\end{aligned}$$

Since $\left\{\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\|\right\}_{n=1}^{\infty}$ is a bounded increasing sequence, $\left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)}\right]_{\nu\lrcorner}\mathbf{x}\right\|_2 \leq \lim_{n\to\infty}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\| = \sup_{n}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\|$. Thus $\left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)}\right]_{\nu\lrcorner}\right\| \leq \sup_{n}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\|$ for all positive integer ν. It follows that $\left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)}\right]\right\| \leq \sup_{n}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\|$. Hence $\sup_{n}\left\|\sum_{k=1}^{n}\left[a_{ji}^{(k)}\right]\right\| = \left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)}\right]\right\|$. □

From the lemma above, the following simple characterization of the set $\mathscr{O}_b(\mathscr{A})$ is immediately obtained.

Theorem 3.2.2. *Let* $\left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{S}^2(\mathscr{A})$. *Then* $\{A_k\}_{k=1}^{\infty} \in \mathscr{O}_b(\mathscr{A})$ *if and only if* $\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\left[\sum_{k=1}^{\infty}\varphi\left(a_{ji}^{(k)*}a_{ji}^{(k)}\right)\right]\right\| < \infty.$

A simple characterization of the set $\mathscr{O}_c(\mathscr{A})$ is obtained as follows.

Theorem 3.2.3. *Let* $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{S}^2(\mathscr{A})$*. Then* $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$ *if and only if the sequences* $\left\{\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right)\right\}_{n=1}^{\infty}$ *converge in* $\mathcal{B}(l_2)$ *uniformly on* $\mathscr{S}(\mathscr{A})$*.*

Proof. Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{S}^2(\mathscr{A})$. Suppose that the sequences $\left\{\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right)\right\}_{n=1}^{\infty}$ converge in $\mathcal{B}(l_2)$ uniformly on $\mathscr{S}(\mathscr{A})$. We will show that $\sum_{k=1}^{\infty}A_k^{[2]}$ converge in $\mathcal{M}_b^{\infty}(\mathscr{A})$. To see this, let $\epsilon > 0$ be given. Then by the assumption, there exists a positive integer N such that for every $\varphi \in \mathscr{S}(\mathscr{A})$,

$$\left\|\varphi\left(\sum_{k=\nu}^{\mu}A_k^{[2]}\right)\right\| = \left\|\sum_{k=\nu}^{\mu}\varphi\left(A_k^{[2]}\right)\right\| < \frac{\epsilon}{2} \text{ for all } \mu > \nu \geq N.$$

So

$$\left\|\sum_{k=\nu}^{\mu}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} = \sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=\nu}^{\mu}A_k^{[2]}\right)\right\| \leq \frac{\epsilon}{2} \text{ for all } \mu > \nu \geq N.$$

It follows that $\left\{\sum_{k=1}^{n}A_k^{[2]}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Thus, by the completeness of $\mathcal{M}_b^{\infty}(\mathscr{A})$, we have $\sum_{k=1}^{\infty}A_k^{[2]}$ converges in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Conversely, suppose that $\mathbf{A}$ belongs to $\mathscr{O}_c(\mathscr{A})$. We will show that the sequence $\left\{\sum_{k=1}^{n}\varphi\left(A^{[2]}\right)\right\}_{n=1}^{\infty}$ converges to $\varphi\left(\sum_{k=1}^{\infty}A_k^{[2]}\right)$ uniformly on $\mathscr{S}(\mathscr{A})$. Let $\epsilon > 0$ be given. Then there exists a positive integer N such that

$$\left\|\sum_{k=1}^{n}A_k^{[2]} - \sum_{k=1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Thus, for any $\varphi \in \mathscr{S}(\mathscr{A})$, we have that

$$\begin{aligned}\left\|\sum_{k=1}^{n}\varphi\left(A^{[2]}\right) - \varphi\left(\sum_{k=1}^{\infty}A_k^{[2]}\right)\right\| &= \left\|\varphi\left(\sum_{k=n+1}^{\infty}A_k^{[2]}\right)\right\| \leq \left\|\sum_{k=n+1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \\ &= \left\|\sum_{k=1}^{n}A_k^{[2]} - \sum_{k=1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \frac{\epsilon}{2} \text{ for all } n \geq N.\end{aligned}$$

The proof is finished. □

From the above characterization of the set $\mathscr{O}_c(\mathscr{A})$, another interesting space of sequences in $\mathcal{S}^2(\mathscr{A})$ arises. Let

$$\mathscr{O}_{pc}(\mathscr{A}) = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2(\mathscr{A}) : \forall \varphi \in \mathscr{S}(\mathscr{A}), \left\{ \sum_{k=1}^{n} \varphi\left(A_k^{[2]}\right) \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{B}(l_2) \right\}.$$

It is obvious by Theorem 3.2.3 that $\mathscr{O}_c(\mathscr{A}) \subseteq \mathscr{O}_{pc}(\mathscr{A})$. The following example shows that the inclusion $\mathscr{O}_c(\mathscr{A}) \subseteq \mathscr{O}_{pc}(\mathscr{A})$ can be proper.

Example 3.2.4. *For each positive integer k, let f_k be a function defined on $[0,1]$ by $f_k(t) = (t^k - t^{k+1})^{1/4}$. It is clear that $f_k \in C[0,1]$ for all k. Let, for each k, A_k be the matrix whose $(1,k)$-th entry is the function f_k and all other entries are 0's. Then the sequence $\{A_k\}_{k=1}^{\infty}$ belongs to $\mathscr{O}_{pc}(\mathscr{A})$ but does not belong to $\mathscr{O}_c(\mathscr{A})$. To see this explicitly, we will show first that $\{\varphi(f_k^2)\}_{k=1}^{\infty} \in l_2$ for all $\varphi \in \mathscr{S}(C[0,1])$. To prove this, let $\varphi \in \mathscr{S}(C[0,1])$ be given. Then for each $\{\lambda_k\}_{k=1}^{\infty} \in l_2$,*

$$\begin{aligned}
\sum_{k=1}^{n} \left|\varphi\left(f_k^2\right)\lambda_k\right| &= \sum_{k=1}^{n} \varphi\left(f_k^2\right)|\lambda_k| = \sum_{k=1}^{n} \varphi\left(|\lambda_k| f_k^2\right) \\
&= \varphi\left(\sum_{k=1}^{n} |\lambda_k| f_k^2\right) \leq \|\varphi\| \left\|\sum_{k=1}^{n} |\lambda_k| f_k^2\right\|_{\infty} \\
&= \left\|\sum_{k=1}^{n} |\lambda_k| f_k^2\right\|_{\infty} = \sup_{t\in[0,1]} \sum_{k=1}^{n} |\lambda_k| \left(t^k - t^{k+1}\right)^{1/2} \\
&\leq \sup_{t\in[0,1]} \left(\sum_{k=1}^{n} |\lambda_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} (t^k - t^{k+1})\right)^{1/2} \\
&= \left(\sum_{k=1}^{n} |\lambda_k|^2\right)^{1/2} \sup_{t\in[0,1]} \left(\sum_{k=1}^{n} (t^k - t^{k+1})\right)^{1/2} \\
&= \left(\sum_{k=1}^{n} |\lambda_k|^2\right)^{1/2} \sup_{t\in[0,1]} \left(t - t^{n+1}\right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{1/2} = \|\{\lambda_k\}_{k=1}^{\infty}\|_2 \quad \text{for all } n.
\end{aligned}$$

This implies that $\{\lambda_k \varphi(f_k^2)\}_{k=1}^{\infty} \in l_1$ for all $\{\lambda_k\}_{k=1}^{\infty} \in l_2$. Thus, by Theorem 2.3.6, the sequence $\{\varphi(f_k^2)\}_{k=1}^{\infty}$ belongs to l_2. Notice that for any matrix A with the first row a sequence $\{\lambda_i\}_{i=1}^{\infty}$ that is an element of l_2 and all other entries 0's, has the norm equal to the norm of the sequence $\{\lambda_i\}_{i=1}^{\infty}$. Indeed, for any $\mathbf{x} = \{\xi_i\}_{i=1}^{\infty} \in l_2$, we have

$$A\mathbf{x} = \left\{ \sum_{i=1}^{\infty} \lambda_i \xi_i, 0, 0, ... \right\}.$$

Thus

$$
\begin{aligned}
\|A\| &= \sup\left\{\|A\mathbf{x}\|_2 : \mathbf{x} \in l_2, \|\mathbf{x}\|_2 \leq 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^{\infty}\lambda_i\xi_i\right| : \mathbf{x} = \{\xi_i\}_{i=1}^{\infty} \in l_2, \|\mathbf{x}\|_2 \leq 1\right\} \\
&= \|\Psi(\{\lambda_i\}_{i=1}^{\infty})\| = \|\{\lambda_i\}_{i=1}^{\infty}\|_2\,,
\end{aligned}
$$

where Ψ *is the usual isometric isomorphism from* l_2 *onto* l_2^*. *Next, we will show that the sequence* $\left\{\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right)\right\}_{n=1}^{\infty}$ *converges in* $\mathcal{B}(l_2)$ *for all* $\varphi \in \mathscr{S}\left(C[0,1]\right)$. *To see this, let* $\varphi \in \mathscr{S}\left(C[0,1]\right)$, *and let* $\epsilon > 0$. *Then by the membership of the sequence* $\{\varphi\left(f_k^2\right)\}_{k=1}^{\infty}$ *in* l_2, *there is a positive integer* N *such that*

$$
\sum_{k=\nu}^{\mu}\left(\varphi\left(f_k^2\right)\right)^2 < \epsilon^2 \text{ for all } \mu > \nu > N.
$$

Consequently,

$$
\begin{aligned}
\left\|\sum_{k=\nu}^{\mu}\varphi\left(A_k^{[2]}\right)\right\| &= \left\|\left\{0, 0, ..., \varphi\left(f_\nu^2\right), \varphi\left(f_{\nu+1}^2\right), ..., \varphi\left(f_\mu^2\right), 0, 0, ...\right\}\right\|_2 \\
&= \left(\sum_{k=\nu}^{\mu}\left(\varphi\left(f_k^2\right)\right)^2\right)^{1/2} < \epsilon \text{ for all } \nu > \mu > N.
\end{aligned}
$$

This yields that $\left\{\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right)\right\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $\mathcal{B}(l_2)$, *and hence it is convergent. Therefore* $\{A_k\}_{k=1}^{\infty} \in \mathscr{O}_{pc}\left(C[0,1]\right)$. *We claim that* $\{A_k\}_{k=1}^{\infty} \notin \mathscr{O}_c\left(C[0,1]\right)$. *For each positive integer* n, *let* δ_n *be the pure state of* $C[0,1]$ *that is the point evaluation at* $1 - \frac{1}{n}$, *i.e.,* δ_n *is defined by* $\delta_n(f) = f\left(1 - \frac{1}{n}\right)$ *for all* $f \in C[0,1]$. *Suppose to the contrary that* $\{A_k\}_{k=1}^{\infty} \in \mathscr{O}_c\left(C[0,1]\right)$. *Then there is a positive integer* N *such that*

$$
\left\|\sum_{k=1}^{n-1}A_k^{[2]} - \sum_{k=1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \sqrt{\frac{\frac{1}{e} - \frac{1}{e^2}}{2}} \text{ for all } n \geq N.
$$

Since $\sum_{k=1}^{\infty}A_k^{[2]}$ *is precisely the matrix whose first row is the sequence* $\{f_k^2\}_{k=1}^{\infty}$ *and all other entries are 0's, it follows that*

$$
\left\{\left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n+1}\right\}^{1/2} = \left\{\sum_{k=n}^{2n}\left(\left(1 - \frac{1}{n}\right)^k - \left(1 - \frac{1}{n}\right)^{k+1}\right)\right\}^{1/2}
$$

$$= \left\{ \sum_{k=n}^{2n} \left(f_k^2 \left(1 - \frac{1}{n} \right) \right)^2 \right\}^{1/2} = \left(\sum_{k=n}^{2n} \left(\delta_n \left(f_k^2 \right) \right)^2 \right)^{1/2}$$

$$\leq \sup_{\varphi \in \mathscr{S}(C[0,1])} \left(\sum_{k=n}^{\infty} \left(\varphi \left(f_k^2 \right) \right)^2 \right)^{1/2}$$

$$= \sup_{\varphi \in \mathscr{S}(C[0,1])} \left\| \{0, 0, ..., 0, \varphi\left(f_n^2\right), \varphi\left(f_{n+1}^2\right), ...\} \right\|_2$$

$$= \sup_{\varphi \in \mathscr{S}(C[0,1])} \left\| \varphi \left(\sum_{k=1}^{n-1} A_k^{[2]} - \sum_{k=1}^{\infty} A_k^{[2]} \right) \right\|$$

$$= \left\| \sum_{k=1}^{n-1} A_k^{[2]} - \sum_{k=1}^{\infty} A_k^{[2]} \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} < \sqrt{\frac{\frac{1}{e} - \frac{1}{e^2}}{2}} \text{ for all } n \geq N.$$

Thus, by taking the limit as $n \to \infty$*, we have* $\sqrt{\frac{1}{e} - \frac{1}{e^2}} \leq \sqrt{\frac{\frac{1}{e} - \frac{1}{e^2}}{2}}$*, which is impossible. Therefore,* $\{A_k\}_{k=1}^{\infty} \notin \mathscr{O}_c(C[0,1])$*.*

From Lemma 3.2.1, we have for each $\left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \in \mathscr{O}_{pc}(\mathscr{A})$ that the matrix $\left[\sum_{k=1}^{\infty} \varphi \left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right]$ belongs to $\mathcal{B}(l_2)$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. Since every $f \in \mathscr{A}^*$ is a linear combination of at most four states and $\mathcal{B}(l_2)$ is a vector space, we have for each $\left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \in \mathscr{O}_{pc}(\mathscr{A})$ that $\left[\sum_{k=1}^{\infty} f \left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right]$ belongs to $\mathcal{B}(l_2)$ for all $f \in \mathscr{A}^*$. For any $f \in \mathscr{A}^*$, if $\sum_{k=1}^{\infty} f \left(A_k^{[2]} \right) = [b_{ji}]$, then $\left\{ \sum_{k=1}^{n} f \left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right\}_{n=1}^{\infty}$ converges to b_{ji} for all (j, i). It follows for each $f \in \mathscr{A}^*$ that

$$\sum_{k=1}^{\infty} \left[f \left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right] = \sum_{k=1}^{\infty} f \left(A_k^{[2]} \right) = \left[\sum_{k=1}^{\infty} f \left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right].$$

Theorem 3.2.5. *For every* $\{A_k\}_{k=1}^{\infty}$ *in* $\mathscr{O}_{pc}(\mathscr{A})$*,* $\sup_{\varphi \in \mathscr{S}(\mathscr{A})} \left\| \sum_{k=1}^{\infty} \varphi \left(A_k^{[2]} \right) \right\| < \infty$*.*

Proof. Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty} \in \mathscr{O}_{pc}(\mathscr{A})$. For each n, we define $\Psi_{\mathbf{A}}^{(n)} : \mathscr{A}^* \to \mathcal{B}(l_2)$ by $\Psi_{\mathbf{A}}^{(n)}(f) = \sum_{k=1}^{n} f \left(A_k^{[2]} \right)$ for all $f \in \mathscr{A}^*$. By the linearity of $f \in \mathscr{A}^*$, we have $\Psi_{\mathbf{A}}^{(n)}$ is linear for all n. Let $f \in \mathscr{A}^*$. Then there exist states φ_1, φ_2, φ_3, φ_4, and complex numbers ν_1, ν_2, ν_3, ν_4 whose absolute values are less than or equal to $\|f\|$ such that $f = \nu_1\varphi_1 + \nu_2\varphi_2 + \nu_3\varphi_3 + \nu_4\varphi_4$. Then we have for each n that

$$\begin{aligned} \left\|\Psi_{\mathbf{A}}^{(n)}(f)\right\| &= \left\|\sum_{i=1}^{4}\nu_i\sum_{k=1}^{n}\varphi_i\left(A_k^{[2]}\right)\right\| \leq \sum_{i=1}^{4}\sum_{k=1}^{n}\left\|\varphi_i\left(A_k^{[2]}\right)\right\| \\ &\leq \|f\|\left(4\sum_{k=1}^{n}\|A_k\|_2^2\right); \end{aligned}$$

and

$$\begin{aligned} \left\|\Psi_{\mathbf{A}}^{(n)}(f)\right\| &= \left\|\sum_{i=1}^{4}\nu_i\sum_{k=1}^{n}\varphi_i\left(A_k^{[2]}\right)\right\| \leq \|f\|\left(\sum_{i=1}^{4}\left\|\sum_{k=1}^{n}\varphi_i\left(A_k^{[2]}\right)\right\|\right) \\ &\leq \|f\|\left(\sum_{i=1}^{4}\left\|\sum_{k=1}^{\infty}\varphi_i\left(A_k^{[2]}\right)\right\|\right). \end{aligned}$$

The first inequality implies that the operator $\Psi_{\mathbf{A}}^{(n)}$ is bounded for all n, and the other one implies that the set $\left\{\left\|\Psi_{\mathbf{A}}^{(n)}(f)\right\| : n = 1, 2, 3, ...\right\}$ is bounded for all $f \in \mathscr{A}^*$. Thus, by the uniform boundedness principle, we have that the set $\left\{\left\|\Psi_{\mathbf{A}}^{(n)}\right\| : n = 1, 2, 3, ...\right\}$ is bounded. Let $M = \sup\left\{\left\|\Psi_{\mathbf{A}}^{(n)}\right\| : n = 1, 2, 3, ...\right\}$, and let $\varphi \in \mathscr{S}(\mathscr{A})$. Then

$$\begin{aligned} \left\|\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| &\leq \left\|\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right) - \sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| + \left\|\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right)\right\| \\ &= \left\|\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right) - \sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| + \left\|\Psi_{\mathbf{A}}^{(n)}(\varphi)\right\| \\ &\leq \left\|\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right) - \sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| + \left\|\Psi_{\mathbf{A}}^{(n)}\right\| \\ &\leq \left\|\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right) - \sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| + M \text{ for all } n. \end{aligned}$$

Thus, by taking the limit as $n \to \infty$, we have for any $\varphi \in \mathscr{S}(\mathscr{A})$ that

$$\left\|\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| \leq M.$$

It follows that

$$\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\| \leq M < \infty.$$

The proof is finished. □

From Theorem 3.2.2 and Theorem 3.2.5, the following corollary is obtained.

Corollary 3.2.6. *$\mathscr{O}_{pc}(\mathscr{A}) \subseteq \mathscr{O}_b(\mathscr{A})$.*

The example below shows that the inclusion $\mathscr{O}_{pc}(\mathscr{A}) \subseteq \mathscr{O}_b(\mathscr{A})$ is always proper.

Example 3.2.7. *For each positive integer k, let A_k be the matrix with (k,k)-th entry the identity I of $\mathscr{A}$ and all other entries 0. For any $\varphi \in \mathscr{S}(\mathscr{A})$, we have $\varphi(I^*I) = \varphi(I^*) = \varphi(I) = \|\varphi\| = 1$. Thus, for each positive integers ν, μ with $\nu > \mu$, the matrix $\displaystyle\sum_{k=\mu}^{\nu} \varphi\left(A_k^{[2]}\right)$ is the scalar matrix whose (k,k)-th entry is 1 for all $\mu \le k \le \nu$ and all other entries are 0's. From this, we obtain for each $\varphi \in \mathscr{S}(\mathscr{A})$ that*

$$\left\| \sum_{k=\mu}^{\nu} \varphi\left(A_k^{[2]}\right) \right\| = 1 \text{ for all } \nu > \mu.$$

This implies for each $\varphi \in \mathscr{S}(\mathscr{A})$ that the sequence $\left\{ \displaystyle\sum_{k=1}^{n} \varphi\left(A_k^{[2]}\right) \right\}_{n=1}^{\infty}$ is a bounded non-Cauchy sequence in $\mathcal{B}(l_2)$. Hence $\{A_k\}_{k=1}^{\infty}$ belongs to $\mathscr{O}_b(\mathscr{A})$ but doesn't belong to $\mathscr{O}_{pc}(\mathscr{A})$.

For any $\mathbf{A} = \left\{ A_k = \left[a_{ji}^{(k)}\right] \right\}_{k=1}^{\infty} \in \mathscr{O}_b(\mathscr{A})$, we define

$$|||\mathbf{A}||| := \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \left\| \left[\sum_{k=1}^{\infty} \varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right] \right\|^{1/2}.$$

Since for each k and $\varphi \in \mathscr{S}(\mathscr{A})$, we have

$$\left\| \varphi\left(A_k^{[2]}\right) \right\| = \left\| \left[\varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right] \right\| \le \left\| \left[\sum_{k=1}^{\infty} \varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right] \right\|,$$

it follows that $|||A_k|||_2 \le |||\mathbf{A}|||$ for all k. For the case where $\mathbf{A} \in \mathscr{O}_{pc}(\mathscr{A})$, we have

$$|||\mathbf{A}||| = \sup_{\varphi \in \mathscr{S}(\mathscr{A})} \left\| \sum_{k=1}^{\infty} \varphi\left(A_k^{[2]}\right) \right\|^{1/2}.$$

In the following proposition, we obtain another form of $|||\mathbf{A}|||$ for any $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$.

Proposition 3.2.8. *If $\mathbf{A} = \{A_k\}_{k=1}^{\infty} \in \mathscr{O}_c(\mathscr{A})$, then $|||\mathbf{A}||| = \left\| \displaystyle\sum_{k=1}^{\infty} A_k^{[2]} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}$.*

Proof. If $\mathbf{A} = \left\{ A_k = \left[a_{ji}^{(k)}\right] \right\}_{k=1}^{\infty} \in \mathscr{O}_c(\mathscr{A})$, then by the preceding theorem, we have $\displaystyle\sum_{k=1}^{\infty} A_k^{[2]}$ converges in $\mathcal{M}_b^{\infty}(\mathscr{A})$. It is easy to see that

$$\sum_{k=1}^{\infty} A_k^{[2]} = \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right].$$

Thus we obtain in this circumstance that

$$
\begin{aligned}
\|\mathbf{A}\| &= \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\|\right)^{1/2} = \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\left[\sum_{k=1}^{\infty}\varphi\left(a_{ji}^{(k)^*}a_{ji}^{(k)}\right)\right]\right\|\right)^{1/2}\\
&= \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\left[\varphi\left(\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right)\right]\right\|\right)^{1/2} = \left(\sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=1}^{\infty}A_k^{[2]}\right)\right\|\right)^{1/2}\\
&= \left\|\sum_{k=1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}.
\end{aligned}
$$

The proof is finished. □

Another space of sequences in $\mathcal{S}^2(\mathscr{A})$ which can reasonably be defined is the following:

$$
\mathscr{O}_\kappa(\mathscr{A}) = \left\{\left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}\subseteq\mathcal{S}^2(\mathscr{A}) : \left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]\in\mathcal{K}(\mathscr{A})\right\}.
$$

A characterization of the set $\mathscr{O}_\kappa(\mathscr{A})$ is obtained as follows.

Theorem 3.2.9. *Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{S}^2(\mathscr{A})$. Then $\mathbf{A}\in\mathscr{O}_\kappa(\mathscr{A})$ if and only if each of the following statements holds:*

(1) $\mathbf{A}$ *belongs to* $\mathscr{O}_c(\mathscr{A})$;

(2) $\varphi\left(A_k^{[2]}\right)$ *is compact for all positive integer k and $\varphi\in\mathscr{S}(\mathscr{A})$, and the map defined by* $\varphi\mapsto\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)$ *from $\mathscr{S}(\mathscr{A})$, equipped with the weak* topology relative to $\mathscr{A}^*$, into $\mathcal{K}$ is continuous.*

Proof. Let $\mathbf{A} = \left\{A_k = \left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}$ be a sequence in $\mathcal{S}^2(\mathscr{A})$. Suppose that $\mathbf{A}$ belongs to $\mathscr{O}_\kappa(\mathscr{A})$. We will show first that the assertion (1) holds. To see this, let $\epsilon > 0$ be given. Then by the membership of the sequence $\mathbf{A}$ in $\mathscr{O}_\kappa(\mathscr{A})$, we have the matrix $\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]$ belongs to $\mathcal{M}_b^{\infty}(\mathscr{A})$ and there exists a positive integer N such that

$$
\left\|\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]_{N\lrcorner} - \left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \frac{\epsilon}{4}.
$$

Since for each $1\le j,i\le N$, we have that the series $\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}$ converges in C^*-algebra $\mathscr{A}$, it follows that there exists a positive integer K_0 such that for each

$1 \leq j, i \leq N$,

$$\left\| \sum_{k=K+1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right\| < \frac{\epsilon}{2N^{3/2}} \text{ for all } K \geq K_0.$$

Thus

$$\begin{aligned}
\left\| \sum_{k=1}^{K} A_k^{[2]} - \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} &= \left\| \left[\sum_{k=1}^{K} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] - \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\leq \left\| \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} - \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\quad + \left\| \left[\sum_{k=1}^{K} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} - \left[\sum_{k=1}^{K} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\quad + \left\| \left[\sum_{k=1}^{K} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} - \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\leq 2 \left\| \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} - \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} + \left\| \left[\sum_{k=K+1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\leq 2 \left\| \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]_{N\lrcorner} - \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\quad + \left\{ \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \left\| \sum_{k=K+1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right\| \right)^2 \right\}^{1/2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } K \geq K_0.
\end{aligned}$$

It follows that $\left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty}$ converges to $\left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right]$ in $\mathcal{M}_b^\infty(\mathscr{A})$. Consequently, $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$. Next, we will show that (2) is satisfied. For any $\varphi \in \mathscr{S}(\mathscr{A})$ and positive integer k, we have

$$\begin{aligned}
\left\| \varphi\left(A_k^{[2]} \right)_{n\lrcorner} - \varphi\left(A_k^{[2]} \right) \right\| &= \left\| \varphi\left(\left(A_k^{[2]} \right)_{n\lrcorner} - A_k^{[2]} \right) \right\| \\
&\leq \left\| \left(A_k^{[2]} \right)_{n\lrcorner} - A_k^{[2]} \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \\
&\leq \left\| \left(\sum_{l=1}^{\infty} A_l^{[2]} \right)_{n\lrcorner} - \sum_{l=1}^{\infty} A_l^{[2]} \right\|_{\mathcal{M}_b^\infty(\mathscr{A})} \to 0.
\end{aligned}$$

This yields that $\varphi\left(A_k^{[2]}\right) \in \mathcal{K}$ for all $\varphi \in \mathscr{S}(\mathscr{A})$ and positive integer k. Since from the statement (1), we have $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$ which is a subset of $\mathscr{O}_{pc}(\mathscr{A})$, it follows that $\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right) = \left[\sum_{k=1}^{\infty}\varphi\left(a_{ji}^{(k)^*}a_{ji}^{(k)}\right)\right]$ for all $\varphi \in \mathscr{S}(\mathscr{A})$. So, by Theorem 3.1.3, we have for any $\varphi \in \mathscr{S}(\mathscr{A})$ by the membership of $\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]$ in $\mathcal{K}(\mathscr{A})$ that $\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right) = \left[\sum_{k=1}^{\infty}\varphi\left(a_{ji}^{(k)^*}a_{ji}^{(k)}\right)\right] = \left[\varphi\left(\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right)\right] = \varphi\left(\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]\right) \in \mathcal{K}$ and the map $\varphi \mapsto \sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right) = \varphi\left(\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]\right)$ from $\mathscr{S}(\mathscr{A})$ along with the weak* topology into $\mathcal{K}$ is continuous. Conversely, suppose that the statements (1) and (2) are satisfied. Then $\sum_{k=1}^{\infty}A_k^{[2]}$ converges in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Since $\sum_{k=1}^{\infty}A_k^{[2]} = \left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{k}\right]$ and (2) is satisfied, we have immediately by Theorem 3.1.3 that the matrix $\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{k}\right]$ belongs to $\mathcal{K}(\mathscr{A})$. □

From the preceding theorem, the following corollary is immediately obtained.

Corollary 3.2.10. $\mathscr{O}_\kappa(\mathscr{A}) \subseteq \mathscr{O}_c(\mathscr{A})$.

The following example shows that the inclusion $\mathscr{O}_\kappa(\mathscr{A}) \subseteq \mathscr{O}_c(\mathscr{A})$ is always proper.

Example 3.2.11. *Let A be the matrix whose each entry in the main diagonal is the identity of $\mathscr{A}$ and all other entries are 0's. Since for a fixed $\varphi \in \mathscr{S}(\mathscr{A})$, we have $\varphi(A)$ is exactly the identity matrix in $\mathcal{B}(l_2)$, it follows that $\varphi(A)$ is not compact. Thus $A \notin \mathcal{K}(\mathscr{A})$. Hence the sequence $\{A_k\}_{k=1}^{\infty}$, where $A_1 = A$ and $A_k = 0$ otherwise, belongs to $\mathscr{O}_c(\mathscr{A})$ but doesn't belong to $\mathscr{O}_\kappa(\mathscr{A})$.*

The remaining contents of this chapter are all about proving the completeness of the four sequence spaces. To see that they are normed spaces, the following Cauchy-Schwarz's inequality, which is an extension of the one in [13], is constructed.

Theorem 3.2.12. (Cauchy-Schwarz's inequality) *For any two n-tuples $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$ of matrices in $\mathcal{S}^2(\mathscr{A})$ and $\varphi \in \mathscr{S}(\mathscr{A})$,*

$$\left\|\sum_{k=1}^{n}\varphi\left(A_k \bullet B_k\right)\right\| \leq \left\|\sum_{k=1}^{n}\varphi\left(A_k^{[2]}\right)\right\|^{1/2}\left\|\sum_{k=1}^{n}\varphi\left(B_k^{[2]}\right)\right\|^{1/2}.$$

Proof. Let $\varphi \in \mathscr{S}(\mathscr{A})$, and suppose that $A_k = \left[a_{ji}^{(k)}\right]$ and $B_k = \left[b_{ji}^{(k)}\right]$ for all $k = 1, ..., n$. Then by Theorem 3.1.10, $\varphi(A_k \bullet B_k) \in \mathcal{B}(l_2)$ for all k. Let $\mathbf{x} = \{x_i\}_{i=1}^{\infty} \in l_2$ with $\|\mathbf{x}\|_2 \leq 1$. Then by Cauchy-Schwarz's inequality for scalar sequences,

$$
\begin{aligned}
\left\|\sum_{k=1}^{n} \varphi(A_k \bullet B_k)\mathbf{x}\right\|^2 &\leq \sum_{j=1}^{\infty} \left|\sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} \left|\varphi\left(a_{ji}^{(k)} b_{ji}^{(k)}\right)\right|\right) x_i\right|^2 \\
&\leq \sum_{j=1}^{\infty} \left\{\sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} \left|\varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)}\right)\right|^{1/2} \left|\varphi\left(b_{ji}^{(k)^*} b_{ji}^{(k)}\right)\right|^{1/2}\right) |x_i|\right\}^2 \\
&\leq \sum_{j=1}^{\infty} \left\{\sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} \left|\varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)}\right)\right|\right)^{1/2} \left(\sum_{k=1}^{n} \left|\varphi\left(b_{ji}^{(k)^*} b_{ji}^{(k)}\right)\right|\right)^{1/2} |x_i|\right\}^2 \\
&\leq \sum_{j=1}^{\infty} \left\{\sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} \left|\varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)}\right)\right| |x_i|\right)^{1/2} \left(\sum_{k=1}^{n} \left|\varphi\left(b_{ji}^{(k)^*} b_{ji}^{(k)}\right)\right| |x_i|\right)^{1/2}\right\}^2 \\
&\leq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty}\sum_{k=1}^{n} \left|\varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)}\right)\right| |x_i|\right) \left(\sum_{i=1}^{\infty}\sum_{k=1}^{n} \left|\varphi\left(b_{ji}^{(k)^*} b_{ji}^{(k)}\right)\right| |x_i|\right) \\
&\leq \left\{\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty}\sum_{k=1}^{n} \left|\varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)}\right)\right| |x_i|\right)^2\right\}^{1/2} \\
&\qquad \times \left\{\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty}\sum_{k=1}^{n} \left|\varphi\left(b_{ji}^{(k)^*} b_{ji}^{(k)}\right)\right| |x_i|\right)^2\right\}^{1/2} \\
&\leq \left\|\sum_{k=1}^{n} \varphi\left(A_k^{[2]}\right) |\mathbf{x}|\right\|_2 \left\|\sum_{k=1}^{n} \varphi\left(B_k^{[2]}\right) |\mathbf{x}|\right\|_2 \\
&\leq \left\|\sum_{k=1}^{n} \varphi\left(A_k^{[2]}\right)\right\| \left\|\sum_{k=1}^{n} \varphi\left(B_k^{[2]}\right)\right\|.
\end{aligned}
$$

It follows that

$$
\left\|\sum_{k=1}^{n} \varphi(A_k \bullet B_k)\right\| \leq \left\|\sum_{k=1}^{n} \varphi\left(A_k^{[2]}\right)\right\|^{1/2} \left\|\sum_{k=1}^{n} \varphi\left(B_k^{[2]}\right)\right\|^{1/2}.
$$

The proof is complete. □

From Cauchy-Schwarz's inequality above, the corresponding Minkowski's inequality, which is also an extension of that in [13], is obtained.

Theorem 3.2.13. (Minkowski's inequality) *For any two n-tuples $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$ of matrices in $\mathcal{S}^2(\mathscr{A})$ and $\varphi \in \mathscr{S}(\mathscr{A})$,*

$$\left\| \sum_{k=1}^{n} \varphi\left((A_k + B_k)^{[2]} \right) \right\|^{1/2} \leq \left\| \sum_{k=1}^{n} \varphi\left(A_k^{[2]} \right) \right\|^{1/2} + \left\| \sum_{k=1}^{n} \varphi\left(B_k^{[2]} \right) \right\|^{1/2}.$$

Proof. From Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \varphi\left((A_k + B_k)^{[2]} \right) \right\| &= \left\| \sum_{k=1}^{n} \varphi((A_k + B_k)^* \bullet (A_k + B_k)) \right\| \\
&= \left\| \sum_{k=1}^{n} \varphi((A_k^* + B_k^*) \bullet (A_k + B_k)) \right\| \\
&= \left\| \sum_{k=1}^{n} \varphi(A_k^* \bullet (A_k + B_k)) + \sum_{k=1}^{n} \varphi(B_k^* \bullet (A_k + B_k)) \right\| \\
&\leq \left\| \sum_{k=1}^{n} \varphi(A_k^* \bullet (A_k + B_k)) \right\| + \left\| \sum_{k=1}^{n} \varphi(B_k^* \bullet (A_k + B_k)) \right\| \\
&\leq \left\| \sum_{k=1}^{n} \varphi\left(A_k^{[2]} \right) \right\|^{1/2} \left\| \sum_{k=1}^{n} \varphi\left((A_k + B_k)^{[2]} \right) \right\|^{1/2} \\
&\qquad + \left\| \sum_{k=1}^{n} \varphi\left(B_k^{[2]} \right) \right\|^{1/2} \left\| \sum_{k=1}^{n} \varphi\left((A_k + B_k)^{[2]} \right) \right\|^{1/2} \\
&= \left(\left\| \sum_{k=1}^{n} \varphi\left(A_k^{[2]} \right) \right\|^{1/2} + \left\| \sum_{k=1}^{n} \varphi\left(B_k^{[2]} \right) \right\|^{1/2} \right) \left\| \sum_{k=1}^{n} \varphi\left((A_k + B_k)^{[2]} \right) \right\|^{1/2},
\end{aligned}$$

where $C^* = [c_{ji}^*]$ for any matrix $C = [c_{ji}]$ over $\mathscr{A}$. It follows that

$$\left\| \sum_{k=1}^{n} \varphi\left((A_k + B_k)^{[2]} \right) \right\|^{1/2} \leq \left\| \sum_{k=1}^{n} \varphi\left(A_k^{[2]} \right) \right\|^{1/2} + \left\| \sum_{k=1}^{n} \varphi\left(B_k^{[2]} \right) \right\|^{1/2}.$$

The proof is complete. □

By Lemma 3.2.1 and the Minkowki's inequality above, the triangle inequality for the norm $|||\cdot|||$ is immediately obtained.

Corollary 3.2.14. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be members of* $\mathscr{O}_b(\mathscr{A})$. *Then* $\mathbf{A} + \mathbf{B} \in \mathscr{O}_b(\mathscr{A})$ *and* $|||\mathbf{A} + \mathbf{B}||| \leq |||\mathbf{A}||| + |||\mathbf{B}|||$.

On the space $\mathscr{O}_{pc}(\mathscr{A})$, Cauchy-Schwarz's inequality and Minkowski's inequality can be extended, by the continuity of the operator norm on $\mathcal{B}(l_2)$, to infinite sums as follows.

Corollary 3.2.15. *Let* $\{A_k\}_{k=1}^{\infty}$ *and* $\{B_k\}_{k=1}^{\infty}$ *be members of* $\mathscr{O}_{pc}(\mathscr{A})$*. Then for any* $\varphi \in \mathscr{S}(\mathscr{A})$*,*

(1) $$\left\|\sum_{k=1}^{\infty}\varphi(A_k \bullet B_k)\right\| \leq \left\|\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\|^{1/2}\left\|\sum_{k=1}^{\infty}\varphi\left(B_k^{[2]}\right)\right\|^{1/2};$$

(2) $$\left\|\sum_{k=1}^{\infty}\varphi\left((A_k + B_k)^{[2]}\right)\right\|^{1/2} \leq \left\|\sum_{k=1}^{\infty}\varphi\left(A_k^{[2]}\right)\right\|^{1/2} + \left\|\sum_{k=1}^{\infty}\varphi\left(B_k^{[2]}\right)\right\|^{1/2}.$$

By Proposition 3.2.8 and the corollary above, the following corollary is immediately obtained.

Corollary 3.2.16. *If* $\{A_k\}_{k=1}^{\infty}$ *and* $\{B_k\}_{k=1}^{\infty}$ *be members of* $\mathscr{O}_c(\mathscr{A})$*, then*

(1) $$\left\|\sum_{k=1}^{\infty}A_k \bullet B_k\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \leq \left\|\sum_{k=1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}\left\|\sum_{k=1}^{\infty}B_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2};$$

(2) $$\left\|\sum_{k=1}^{\infty}(A_k + B_k)^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \leq \left\|\sum_{k=1}^{\infty}A_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} + \left\|\sum_{k=1}^{\infty}B_k^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}.$$

We are now ready to prove the Rieze-Fisher-type theorem for completeness of the four sequence spaces.

Theorem 3.2.17. *The four sets* $\mathscr{O}_b(\mathscr{A})$*,* $\mathscr{O}_{pc}(\mathscr{A})$*,* $\mathscr{O}_c(\mathscr{A})$*, and* $\mathscr{O}_\kappa(\mathscr{A})$ *equipped with the norm* $|||\cdot|||$ *are Banach spaces. Indeed, the three sequence spaces* $\mathscr{O}_{pc}(\mathscr{A})$*,* $\mathscr{O}_c(\mathscr{A})$*, and* $\mathscr{O}_\kappa(\mathscr{A})$ *are all closed subspaces of* $\mathscr{O}_b(\mathscr{A})$*.*

Proof. First, we will show that $\mathscr{O}_b(\mathscr{A})$ endowed with the norm $|||\cdot|||$ is a normed space. For any $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ in $\mathscr{O}_b(\mathscr{A})$, and $\alpha \in \mathbb{C}$, it is clear that $\alpha\mathbf{A}$ belongs to $\mathscr{O}_b(\mathscr{A})$. By Corollary 3.2.14, we have that the set $\mathscr{O}_b(\mathscr{A})$ is closed under the addition. Thus the set $\mathscr{O}_b(\mathscr{A})$ is a vector space. It is clear that $|||\mathbf{A}||| \geq 0$, $|||\alpha\mathbf{A}||| = |\alpha|\,|||\mathbf{A}|||$ and $|||0||| = 0$. If $|||\mathbf{A}||| = 0$, then $|||A_k|||_2 = 0$ for all k, which implies that $A_k = 0$ for all k. Thus $\mathbf{A} = 0$. By Corollary 3.2.14 again, the triangle inequality for the norm $|||\cdot|||$ has obtained. Thus $\mathscr{O}_b(\mathscr{A})$ equipped with the norm $|||\cdot|||$ is a normed space. To see that $\mathscr{O}_b(\mathscr{A})$ is a Banach space, let $\left\{\mathbf{A}_n = \left\{A_k^{(n)}\right\}_{k=1}^{\infty}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathscr{O}_b(\mathscr{A})$. For each k, we have

$$\left|\left|\left|A_k^{(n)} - A_k^{(m)}\right|\right|\right|_2 \leq |||\mathbf{A}_n - \mathbf{A}_m||| \text{ for all } n, m.$$

So $\left\{A_k^{(n)}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{S}^2(\mathscr{A})$ for all k. Thus, by the completeness of $\mathcal{S}^2(\mathscr{A})$, we have for each k that there is A_k in $\mathcal{S}^2(\mathscr{A})$ such that $A_k^{(n)} \to A_k$. Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$, we will show that $\mathbf{A} \in \mathscr{O}_b(\mathscr{A})$ and $\mathbf{A}_n \to \mathbf{A}$. In order to prove these, let $\epsilon > 0$ be given. Then there exists a positive integer N such that

$$|||\mathbf{A}_n - \mathbf{A}_m||| \leq \frac{\epsilon}{2} \text{ for all } n, m \geq N.$$

Let K be a positive integer. Then we have

$$\left\|\sum_{k=1}^{K}\left(A_k^{(n)}-A_k^{(m)}\right)^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \leq \|\mathbf{A}_n-\mathbf{A}_m\|^2 \leq \frac{\epsilon^2}{4} \text{ for all } n,m \geq N. \qquad (*)$$

Since for each k, we have $\left\{A_k^{(m)}\right\}_{m=1}^{\infty}$ converges to A_k in $\mathcal{S}^2(\mathscr{A})$, it follows for each fixed k and n that $\left\{A_k^{(n)}-A_k^{(m)}\right\}_{m=1}^{\infty}$ converges to $A_k^{(n)}-A_k$ in $\mathcal{S}^2(\mathscr{A})$. Hence, by Proposition 3.1.9, we have for each fixed k and n that $\left\{\left(A_k^{(n)}-A_k^{(m)}\right)^{[2]}\right\}_{m=1}^{\infty}$ converges to $\left(A_k^{(n)}-A_k\right)^{[2]}$ in $\mathcal{M}_b^{\infty}(\mathscr{A})$ for all k. Hence, for each fixed n, we obtain that $\left\{\sum_{k=1}^{K}\left(A_k^{(n)}-A_k^{(m)}\right)^{[2]}\right\}_{m=1}^{\infty}$ converges to $\sum_{k=1}^{K}\left(A_k^{(n)}-A_k\right)^{[2]}$ in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Thus, by taking the limits as $m\to\infty$ on both sides of $(*)$, we have by the continuity of the operator norm on $\mathcal{M}_b^{\infty}(\mathscr{A})$ that

$$\left\|\sum_{k=1}^{K}\left(A_k^{(n)}-A_k\right)^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \leq \frac{\epsilon^2}{4} \text{ for all } n \geq N. \qquad (**)$$

Sine K was given arbitrarily, we have by $(**)$ that

$$\|\mathbf{A}_n-\mathbf{A}\| \leq \frac{\epsilon}{2} \text{ for all } n \geq N. \qquad (***)$$

By $(***)$, we obtain that $\mathbf{A}_N-\mathbf{A}$ belongs to $\mathscr{O}_b(\mathscr{A})$, which implies that $\mathbf{A}=\mathbf{A}_N-(\mathbf{A}_N-\mathbf{A})\in\mathscr{O}_b(\mathscr{A})$. From this and $(***)$ again, we obtain that $\mathbf{A}_n\to\mathbf{A}$. Thus $\mathscr{O}_b(\mathscr{A})$ is a Banach space.

To see that $\mathscr{O}_{pc}(\mathscr{A})$ is a Banach space, it suffices to show that $\mathscr{O}_{pc}(\mathscr{A})$ is a closed subspace of $\mathscr{O}_b(\mathscr{A})$. Let $\mathbf{A}=\{A_k\}_{k=1}^{\infty}$ and $\mathbf{B}=\{B_k\}_{k=1}^{\infty}$ be members of $\mathscr{O}_{pc}(\mathscr{A})$, and $\alpha\in\mathbb{C}$. It is obvious that $\alpha\mathbf{A}$ belongs to $\mathscr{O}_{pc}(\mathscr{A})$. For every $\varphi\in\mathscr{S}(\mathscr{A})$, we have for each $\mu>\nu>1$ by Minkowski's inequality that

$$\left\|\sum_{k=\nu}^{\mu}\varphi\left((A_k+B_k)^{[2]}\right)\right\|^{1/2} \leq \left\|\sum_{k=\nu}^{\mu}\varphi\left(A_k^{[2]}\right)\right\|^{1/2}+\left\|\sum_{k=\nu}^{\mu}\varphi\left(B_k^{[2]}\right)\right\|^{1/2}.$$

Since $\mathbf{A}$ and $\mathbf{B}$ are members of $\mathscr{O}_{pc}(\mathscr{A})$, it follows for each $\varphi\in\mathscr{S}(\mathscr{A})$ that $\left\{\sum_{k=1}^{n}\varphi\left((A_k+B_k)^{[2]}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}(l_2)$. Thus, by the completeness of $\mathcal{B}(l_2)$, the sequence $\left\{\sum_{k=1}^{n}\varphi\left((A_k+B_k)^{[2]}\right)\right\}_{n=1}^{\infty}$ is convergent for all $\varphi\in\mathscr{S}(\mathscr{A})$. Hence $\mathbf{A}+\mathbf{B}\in\mathscr{O}_{pc}(\mathcal{A})$. Therefore, the set $\mathscr{O}_{pc}(\mathcal{A})$ is a vector subspace of $\mathscr{O}_b(\mathcal{A})$. Next, to see that $\mathscr{O}_{pc}(\mathcal{A})$ is closed in $\mathscr{O}_b(\mathcal{A})$, we suppose that $\left\{\mathbf{A}_n=\left\{A_k^{(n)}\right\}_{k=1}^{\infty}\right\}_{n=1}^{\infty}$ is a sequence in $\mathscr{O}_{pc}(\mathscr{A})$ converging to an element

$\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ in $\mathscr{O}_b(\mathscr{A})$. We will show that $\mathbf{A} \in \mathscr{O}_{pc}(\mathscr{A})$. Let $\epsilon > 0$ and $\varphi \in \mathscr{S}(\mathscr{A})$. Then there exists a positive integer N such that

$$\|\!|\mathbf{A}_N - \mathbf{A}|\!\| < \frac{\sqrt{\epsilon}}{2}.$$

Since $\mathbf{A}_N \in \mathscr{O}_{pc}(\mathscr{A})$, there exists a positive integer K such that

$$\left\| \sum_{k=\nu}^{\mu} \varphi\left(\left(A_k^{(N)}\right)^{[2]}\right) \right\| < \frac{\epsilon}{4} \text{ for all } \mu > \nu > K.$$

Thus, by Minkowski's inequality, we have for every $\mu > \nu > K$ that

$$\begin{aligned}
\left\| \sum_{k=\nu}^{\mu} \varphi\left(A_k^{[2]}\right) \right\| &= \left\| \sum_{k=\nu}^{\mu} \varphi\left(\left(\left(A_k^{(N)} - A_k\right) - A_k^{(N)}\right)^{[2]}\right) \right\| \\
&\leq \left(\left\| \sum_{k=\nu}^{\mu} \varphi\left(\left(A_k^{(N)} - A_k\right)^{[2]}\right) \right\|^{1/2} + \left\| \sum_{k=\nu}^{\mu} \varphi\left(\left(A_k^{(N)}\right)^{[2]}\right) \right\|^{1/2} \right)^2 \\
&\leq \left(\|\!|\mathbf{A}_N - \mathbf{A}|\!\| + \frac{\sqrt{\epsilon}}{2} \right)^2 < \left(\frac{\sqrt{\epsilon}}{2} + \frac{\sqrt{\epsilon}}{2} \right)^2 = \epsilon.
\end{aligned}$$

It follows that $\mathbf{A} \in \mathscr{O}_{pc}(\mathscr{A})$. Therefore $\mathscr{O}_{pc}(\mathscr{A})$ is a closed subspace of $\mathscr{O}_b(\mathscr{A})$, which implies that $\mathscr{O}_{pc}(\mathscr{A})$ is a Banach space.

To see that $\mathscr{O}_c(\mathscr{A})$ is a Banach space, it suffices to show that $\mathscr{O}_c(\mathscr{A})$ is a closed subspace of $\mathscr{O}_{pc}(\mathscr{A})$. Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ and $\mathbf{B} = \{B_k\}_{k=1}^{\infty}$ be members of $\mathscr{O}_c(\mathscr{A})$ and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$. For each $\varphi \in \mathscr{S}(\mathscr{A})$, we have for each $\mu > \nu > 1$ by Minkowski's inequality that

$$\left\| \sum_{k=\nu}^{\mu} \varphi\left((A_k + B_k)^{[2]}\right) \right\|^{1/2} \leq \left\| \sum_{k=\nu}^{\mu} \varphi\left(A_k^{[2]}\right) \right\|^{1/2} + \left\| \sum_{k=\nu}^{\mu} \varphi\left(B_k^{[2]}\right) \right\|^{1/2}.$$

Hence we obtain by taking suprema, as φ runs over the set $\mathscr{S}(\mathscr{A})$, on both sides of the inequality above that

$$\left\|\!\left| \sum_{k=\nu}^{\mu} (A_k + B_k)^{[2]} \right|\!\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \leq \left\|\!\left| \sum_{k=\nu}^{\mu} A_k^{[2]} \right|\!\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} + \left\|\!\left| \sum_{k=\nu}^{\mu} B_k^{[2]} \right|\!\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \quad \text{for all } \mu > \nu.$$

Since $\mathbf{A}, \mathbf{B} \in \mathscr{O}_c(\mathscr{A})$, it follows that $\left\{ \sum_{k=1}^{n} (A_k + B_k)^{[2]} \right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}_b^{\infty}(\mathscr{A})$. Therefore, by the completeness of $\mathcal{M}_b^{\infty}(\mathscr{A})$, the sequence $\left\{ \sum_{k=1}^{n} (A_k + B_k)^{[2]} \right\}_{n=1}^{\infty}$ is convergent. Hence $\mathbf{A} + \mathbf{B} \in \mathscr{O}_c(\mathcal{A})$. Thus the set $\mathscr{O}_c(\mathcal{A})$ is a vector subspace of $\mathscr{O}_{pc}(\mathcal{A})$. We will show that $\mathscr{O}_c(\mathcal{A})$ is closed in $\mathscr{O}_{pc}(\mathcal{A})$. To see this, suppose that $\left\{\mathbf{A}_n = \left\{A_k^{(n)}\right\}_{k=1}^{\infty}\right\}_{n=1}^{\infty}$ is a sequence in $\mathscr{O}_c(\mathscr{A})$ converging to an element $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ in $\mathscr{O}_{pc}(\mathscr{A})$. To get that $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$, let $\epsilon > 0$ be given. Then there exists a positive integer N such that

$$\|\!|\mathbf{A}_N - \mathbf{A}|\!\| < \frac{\sqrt{\epsilon}}{2}.$$

Since $\mathbf{A}_N \in \mathscr{O}_c(\mathscr{A})$, there exists a positive integer K such that

$$\left\|\sum_{k=\nu}^{\mu}\left(A_k^{(N)}\right)^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} < \frac{\epsilon}{4} \quad \text{for all } \mu > \nu > K.$$

Thus, by Mikowski's inequality, we have that

$$\begin{aligned}
\left\|\sum_{k=\nu}^{\mu}A_k^{[2]}\right\|_{\mathcal{M}(\mathcal{A})}^{1/2} &= \sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=\nu}^{\mu}A_k^{[2]}\right)\right\|^{1/2} \\
&\le \sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=\nu}^{\mu}\left(\left(A_k^{(N)} - A_k\right)^{[2]}\right)\right)\right\|^{1/2} \\
&\quad + \sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=\nu}^{\mu}\left(A_k^{(N)}\right)^{[2]}\right)\right\|^{1/2} \\
&\le \sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=1}^{\infty}\left(\left(A_k^{(N)} - A_k\right)^{[2]}\right)\right)\right\|^{1/2} \\
&\quad + \sup_{\varphi\in\mathscr{S}(\mathscr{A})}\left\|\varphi\left(\sum_{k=\nu}^{\mu}\left(A_k^{(N)}\right)^{[2]}\right)\right\|^{1/2} \\
&= \|\!|\mathbf{A}_N - \mathbf{A}|\!\| + \left\|\sum_{k=\nu}^{\mu}\left(A_k^{(N)}\right)^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \\
&< \frac{\sqrt{\epsilon}}{2} + \frac{\sqrt{\epsilon}}{2} = \sqrt{\epsilon} \ \text{ for all } \mu > \nu > K.
\end{aligned}$$

It follows that $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$. Therefore $\mathscr{O}_c(\mathscr{A})$ is a closed subspace of $\mathscr{O}_{pc}(\mathscr{A})$, which implies that $\mathscr{O}_c(\mathscr{A})$ is a Banach space.

Finally, we will show that $\mathscr{O}_\kappa(\mathscr{A})$ is a closed subspace of $\mathscr{O}_c(\mathscr{A})$. Let $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ and $\mathbf{B} = \{B_k\}_{k=1}^{\infty}$ be elements of $\mathscr{O}_\kappa(\mathscr{A})$ and $\alpha \in \mathbb{C}$. It is obvious that $\alpha\mathbf{A} \in \mathscr{O}_\kappa(\mathscr{A})$. Since $\mathbf{A}, \mathbf{B} \in \mathscr{O}_\kappa(\mathscr{A})$ and $\mathscr{O}_\kappa(\mathscr{A}) \subseteq \mathscr{O}_c(\mathscr{A})$, we have $\sum_{k=1}^{\infty}(A_k+B_k)^{[2]}$ converges in $\mathcal{M}_b^{\infty}(\mathscr{A})$ and $\sum_{k=1}^{\infty}(A_k + B_k)^{[2]} = \left[\sum_{k=1}^{\infty}\left(a_{ji}^{(k)} + b_{ji}^{(k)}\right)^*\left(a_{ji}^{(k)} + b_{ji}^{(k)}\right)\right]$. It follows for any positive integer n by Theorem 3.2.16 that

$$\left\|\sum_{k=1}^{\infty}(A_k+B_k)^{[2]}-\left(\sum_{k=1}^{\infty}(A_k+B_k)^{[2]}\right)_{n_{\lrcorner}}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}$$

$$= \left\|\sum_{k=1}^{\infty}((A_k+B_k)-(A_k+B_k)_{n_{\lrcorner}})^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}$$

$$= \left\|\sum_{k=1}^{\infty}((A_k-(A_k)_{n_{\lrcorner}})+(B_k-(B_k)_{n_{\lrcorner}}))^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}$$

$$\leq \left\|\sum_{k=1}^{\infty}((A_k-(A_k)_{n_{\lrcorner}})^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}+\left\|\sum_{k=1}^{\infty}(B_k-(B_k)_{n_{\lrcorner}}))^{[2]}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}$$

$$= \left\|\sum_{k=1}^{\infty}(A_k)^{[2]}-\left(\sum_{k=1}^{\infty}(A_k)^{[2]}\right)_{n_{\lrcorner}}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}$$

$$+\left\|\sum_{k=1}^{\infty}(B_k)^{[2]}-\left(\sum_{k=1}^{\infty}(B_k)^{[2]}\right)_{n_{\lrcorner}}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}.$$

Since $\mathbf{A},\mathbf{B}\in\mathscr{O}_{\kappa}(\mathscr{A})$, by taking the limits as $n\to\infty$ on both sides of the above inequality, we have

$$\left\|\sum_{k=1}^{\infty}(A_k+B_k)^{[2]}-\left(\sum_{k=1}^{\infty}(A_k+B_k)^{[2]}\right)_{n_{\lrcorner}}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2}\to 0.$$

Thus $\mathbf{A}+\mathbf{B}=\{A_k+B_k\}_{k=1}^{\infty}\in\mathscr{O}_{\kappa}(\mathscr{A})$. Hence $\mathscr{O}_{\kappa}(\mathscr{A})$ is a vector subspace of $\mathscr{O}_c(\mathscr{A})$. Next, suppose that $\left\{\mathbf{A}_n=\left\{A_k^{(n)}\right\}_{k=1}^{\infty}\right\}_{n=1}^{\infty}$ is a sequence in $\mathscr{O}_{\kappa}(\mathscr{A})$ converging to an element $\mathbf{A}=\{A_k\}_{k=1}^{\infty}$ in $\mathscr{O}_c(\mathscr{A})$. We will show that $\mathbf{A}\in\mathscr{O}_{\kappa}(\mathscr{A})$. Since $\mathbf{A}\in\mathscr{O}_c(\mathscr{A})$, we have $\sum_{k=1}^{\infty}A_k^{[2]}$ converges in $\mathcal{M}_b^{\infty}(\mathscr{A})$ and $\sum_{k=1}^{\infty}A_k^{[2]}=\left[\sum_{k=1}^{\infty}a_{ji}^{(k)^*}a_{ji}^{(k)}\right]$. To see that $\left\|\sum_{k=1}^{\infty}A_k^{[2]}-\left(\sum_{k=1}^{\infty}A_k^{[2]}\right)_{n_{\lrcorner}}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}\to 0$, let $\epsilon>0$ be given. Then there exists a positive integer N such that

$$\|\mathbf{A}_N-\mathbf{A}\|<\frac{\epsilon}{3}.$$

Since $\mathbf{A}_N\in\mathscr{O}_{\kappa}(\mathscr{A})$, there exists a positive integer J such that

$$\left\|\sum_{k=1}^{\infty}\left(A_k^{(N)}\right)^{[2]}-\left(\sum_{k=1}^{\infty}\left(A_k^{(N)}\right)^{[2]}\right)_{n_{\lrcorner}}\right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}<\left(\frac{\epsilon}{3}\right)^2 \quad \text{for all } n\geq J.$$

It follows that

$$
\begin{aligned}
&\left\| \sum_{k=1}^{\infty} A_k^{[2]} - \left(\sum_{k=1}^{\infty} A_k^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \\
&= \left\| \sum_{k=1}^{\infty} \left(A_k - A_k^{(N)} + A_k^{(N)} \right)^{[2]} - \left(\sum_{k=1}^{\infty} \left(A_k - A_k^{(N)} + A_k^{(N)} \right)^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \\
&\leq \left\| \sum_{k=1}^{\infty} \left(A_k^{(N)} \right)^{[2]} - \left(\sum_{k=1}^{\infty} \left(A_k^{(N)} \right)^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \\
&\qquad + \left\| \sum_{k=1}^{\infty} \left(A_k^{(N)} - A_k \right)^{[2]} - \left(\sum_{k=1}^{\infty} \left(A_k^{(N)} - A_k \right)^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \\
&\leq \left\| \sum_{k=1}^{\infty} \left(A_k^{(N)} \right)^{[2]} - \left(\sum_{k=1}^{\infty} \left(A_k^{(N)} \right)^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} \\
&\qquad + \left(\left\| \sum_{k=1}^{\infty} \left(A_k^{(N)} - A_k \right)^{[2]} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} + \left\| \left(\sum_{k=1}^{\infty} \left(A_k^{(N)} - A_k \right)^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})} \right)^{1/2} \\
&\leq \left\| \sum_{k=1}^{\infty} \left(A_k^{(N)} \right)^{[2]} - \left(\sum_{k=1}^{\infty} \left(A_k^{(N)} \right)^{[2]} \right)_{n_\lrcorner} \right\|_{\mathcal{M}_b^{\infty}(\mathscr{A})}^{1/2} + \sqrt{2}\, \| \mathbf{A}_N - \mathbf{A} \| \\
&< \frac{\epsilon}{3} + \frac{\sqrt{2}\epsilon}{3} = \frac{(1+\sqrt{2})\epsilon}{3} < \epsilon \quad \text{for all } n \geq J.
\end{aligned}
$$

Hence $\mathbf{A} \in \mathscr{O}_\kappa(\mathscr{A})$. The proof is complete. □

Chapter 4

Conclusion

In this thesis, we generalized some results of J. Rakbud and S.-C. Ong in [13] to the setting of matrices over commutative C^*-algebras with identity. Let $\mathscr{A}$ be a commutative C^*-algebra with identity, and let $\mathscr{S}(\mathscr{A})$ be the set of all states of $\mathscr{A}$. Let $\mathcal{M}^\infty(\mathscr{A})$ be the vector space of all infinite matrices over $\mathscr{A}$. For any $A = [a_{ji}] \in \mathcal{M}^\infty(\mathscr{A})$ and $f \in \mathscr{A}^*$, let $f(A) = [f(a_{ji})]$ and $A^{[2]} = \left[a_{ji}^* a_{ji}\right]$. The following are what we have done.

- We defined the class $\mathcal{M}_b^\infty(\mathscr{A})$ as follows:

  $$\mathcal{M}_b^\infty(\mathscr{A}) = \{A \in \mathcal{M}^\infty(\mathscr{A}) : \varphi(A) \in \mathcal{B}(l_2) \quad \text{for all } \varphi \in \mathscr{S}(\mathscr{A})\}$$

  for playing the role as the Banach space $\mathcal{B}(l_2)$ in the setting of Rakbud and Ong. By the closed graph theorem, we obtained for any $A \in \mathcal{M}_b^\infty(\mathscr{A})$ that the quantity $\sup\limits_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A)\|$ is finite. This provided us with a reasonable way of defining a norm on $\mathcal{M}_b^\infty(\mathscr{A})$, and then we obtained that the class $\mathcal{M}_b^\infty(\mathscr{A})$ equipped with the norm defined by $|||A|||_{\mathcal{M}_b^\infty(\mathscr{A})} = \sup\limits_{\varphi \in \mathscr{S}(\mathscr{A})} \|\varphi(A)\|$ is a Banach space.

- We defined the class $\mathcal{K}(\mathscr{A})$ of compact-like matrices as follows:

  $$\mathcal{K}(\mathscr{A}) = \left\{A \in \mathcal{M}_b^\infty(\mathscr{A}) : |||A - A_{n\lrcorner}|||_{\mathcal{M}_b^\infty(\mathscr{A})} \to 0\right\}.$$

  For the case where $\mathscr{A} = \mathbb{C}$, the class $\mathcal{K}(\mathbb{C})$ which is denoted by just $\mathcal{K}$ is exactly the class of all compact matrices regarded as operators on l_2. We obtained some characterizations of $\mathcal{K}(\mathscr{A})$ as follows.

Theorem 4.1 *Let $A \in \mathcal{M}^\infty(\mathscr{A})$. Then the following are equivalent.*

(1) *The matrix A belongs to $\mathcal{K}(\mathscr{A})$.*

(2) *There is a sequence $\{F_n\}_{n=1}^\infty$ of matrices in $\mathcal{M}^\infty(\mathscr{A})$ with finitely many non-zero entries such that $|||F_n - A|||_{\mathcal{M}_b^\infty(\mathscr{A})} \to 0$.*

(3) *The matrix $\varphi(A)$ belongs to $\mathcal{K}$ for all $\varphi \in \mathscr{S}(\mathscr{A})$ and the map $\varphi \mapsto \varphi(A)$ from $\mathscr{S}(\mathscr{A})$ equipped with the topology relative to the weak* topology on $\mathscr{A}^*$ into $\mathcal{K}$ is continuous.*

- We studied the class $\mathcal{S}^2(\mathscr{A})$ of all matrices $A = [a_{ji}] \in \mathcal{M}^{\infty}(\mathscr{A})$ such that $\varphi\left(A^{[2]}\right) \in \mathcal{B}(l_2)$ for all $\varphi \in \mathscr{S}(\mathscr{A})$, which was first defined in [4].
- We defined three sequence spaces by a way analogous to the ones given in [13] as follows:

$$\mathscr{O}_b(\mathscr{A}) = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2(\mathscr{A}) : \left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty} \text{ is bounded in } \mathcal{M}_b^{\infty}(\mathscr{A}) \right\};$$

$$\mathscr{O}_c(\mathscr{A}) = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2(\mathscr{A}) : \left\{ \sum_{k=1}^{n} A_k^{[2]} \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{M}_b^{\infty}(\mathscr{A}) \right\};$$

$$\mathscr{O}_\kappa(\mathscr{A}) = \left\{ \left\{ \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty} \subseteq \mathcal{S}^2(\mathscr{A}) : \left[\sum_{k=1}^{\infty} a_{ji}^{(k)^*} a_{ji}^{(k)} \right] \in \mathcal{K}(\mathscr{A}) \right\},$$

and provided some characterizations of these three spaces as follows.

Theorem 4.2 *Let* $\left\{ A_k = \left[a_{ji}^{(k)} \right] \right\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{S}^2(\mathscr{A})$*. Then* $\{A_k\}_{k=1}^{\infty} \in \mathscr{O}_b(\mathscr{A})$ *if and only if* $\sup_{\varphi \in \mathscr{S}(\mathscr{A})} \left\| \left[\sum_{k=1}^{\infty} \varphi\left(a_{ji}^{(k)^*} a_{ji}^{(k)} \right) \right] \right\| < \infty.$

Theorem 4.3 *Let* $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{S}^2(\mathscr{A})$*. Then* $\mathbf{A} \in \mathscr{O}_c(\mathscr{A})$ *if and only if the sequences* $\left\{ \sum_{k=1}^{n} \varphi\left(A_k^{[2]} \right) \right\}_{n=1}^{\infty}$ *converge in* $\mathcal{B}(l_2)$ *uniformly on* $\mathscr{S}(\mathscr{A})$*.*

Theorem 4.4 *Let* $\mathbf{A} = \{A_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{S}^2(\mathscr{A})$*. Then* $\mathbf{A} \in \mathscr{O}_\kappa(\mathscr{A})$ *if and only if each of the following statements holds:*

(1) $\mathbf{A}$ *belongs to* $\mathscr{O}_c(\mathscr{A})$*;*

(2) $\varphi\left(A_k^{[2]} \right)$ *is compact for all positive integer* k *and* $\varphi \in \mathscr{S}(\mathscr{A})$*, and the map defined by* $\varphi \mapsto \sum_{k=1}^{\infty} \varphi\left(A_k^{[2]} \right)$ *from* $\mathscr{S}(\mathscr{A})$*, equipped with the weak* topology relative to* $\mathscr{A}^*$*, into* $\mathcal{K}$ *is continuous.*

From the characterization of the set $\mathscr{O}_c(\mathscr{A})$, we obtained one more interesting sequence space as follows:

$$\mathscr{O}_{pc}(\mathscr{A}) = \left\{ \{A_k\}_{k=1}^{\infty} \subseteq \mathcal{S}^2(\mathscr{A}) : \forall \varphi \in \mathscr{S}(\mathscr{A}), \left\{ \sum_{k=1}^{n} \varphi\left(A_k^{[2]} \right) \right\}_{n=1}^{\infty} \text{ converges in } \mathcal{B}(l_2) \right\}.$$

The inclusion relations among the four sequence spaces were studied. The following is the result: $\mathscr{O}_{\kappa}(\mathscr{A}) \subsetneq \mathscr{O}_{c}(\mathscr{A}) \subsetneq \mathscr{O}_{pc}(\mathscr{A}) \subsetneq \mathscr{O}_{b}(\mathscr{A})$. Finally, we proved the Rieze-Fisher-type theorem for completeness of our sequence spaces under the norm defined by

$$\left\|\left\{\left[a_{ji}^{(k)}\right]\right\}_{k=1}^{\infty}\right\| = \sup_{\varphi\in\mathscr{S}(\mathscr{A})} \left\|\left[\sum_{k=1}^{\infty}\varphi\left(a_{ji}^{(k)^*}a_{ji}^{(k)}\right)\right]\right\|^{1/2}.$$

In [13], the authors also studied the sequential convergence as well as the duality of their sequence spaces. These interesting problems will be a research topic for us to work on in the future.

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APPENDIX

List of Symbols

$\mathcal{B}(l_2)$	set of all bounded linear operator on l_2
$\mathcal{K}$	set of all compact operator on l_2
$\mathscr{S}(\mathscr{A})$	set of all states of $\mathscr{A}$
$\mathcal{M}(\mathscr{A})$	set of all infinite matrices A over $\mathscr{A}$ such that $\varphi(A) \in \mathcal{B}(l_2)$ for all $\varphi \in \mathscr{S}(\mathscr{A})$
$\mathcal{K}(\mathscr{A})$	set of all matrices A in $\mathcal{M}(\mathscr{A})$ such that $\|\|\|A - A_{n\lrcorner}\|\|\| \to 0$
$\mathcal{S}^2(\mathscr{A})$	set of all infinite matrices A such that $A^{[2]} \in \mathcal{M}(\mathscr{A})$
$A_{n\lrcorner}$	the matrix whose entries in the upper left $n \times n$-block are exactly those of A and are zero otherwise.

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